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Real Quaternionic Calculus Handbook

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Preface

Real Quaternionic Analysis is a multifaceted subject. Created to describe some phenomena in special relativity, electrodynamics, spin, etc. It has developed into a body of material that interacts with many branches of mathematics, such as complex analysis, harmonic analysis, differential geometry, differential equations, as well as into a ubiquitous factor in the description and elucidation of problems in mathematical physics. Quaternions have been towards Maxwell's equations. In the meantime, real quaternionic analysis has become a well-established branch of mathematics and greatly successful in many different directions. Quaternions have been successfully applied to signal processing, most notably pattern recognition. They can be used for image segmentation, finding structure based not only upon color, but repeating patterns. Quaternions may also be used to simplify derivations in computer vision and robotics, to develop computer applications in virtual reality, and so on.

This book is intended to provide material for an introductory one- or two-semester undergraduate course on some of the major aspects of real quaternionic analysis, with exercises. Alternatively, it may be used in a beginning graduate level course and as a reference. That it is why, rather than general theorems, we supply concrete examples and exercises which form the basis of this book. The exercises proposed at the end of each chapter are an essential part of it. The writing herein is straightforward and it is addressed to readers who have no prior knowledge of this subject and who have a basic graduate mathematics background: real and complex analysis, ordinary differential equations, partial differential equations and theory of distributions.

The detailed reference list proposed at the end is seen as an initial point for the development of the topics covered in the handbook. From a reader's point of view, we chose to present it at the end rather than throughout the text.

Here is a brief description of the topics covered in the first ten chapters.

Chapter 1 An introduction to and historical background on the discovery of the quaternions are provided. The definitions and general properties of quaternions are examined in detail.

Chapter 2 The notions of quaternion and spatial rotation are mastered. Some applications of quaternions to plane geometry are mentioned.

- Chapter 3 Studies sequences of quaternion numbers and their properties.
- Chapter 4 Reviews the basic properties of quaternion power series and infinite products.
- Chapter 5 The quaternion exponential, logarithmic and power functions are covered. A brief discussion on the notions of multiple-valued functions and branches is also presented.
- Chapter 6 The quaternion trigonometric functions are defined.
- Chapter 7 The quaternion hyperbolic functions are introduced.
- Chapter 8 The main focus here is on the study of the inverses of the quaternion trigonometric and hyperbolic functions, and their properties.
- Chapter 9 Matrices with quaternion entries are presented. In spite of the difficulties caused by the noncommutativity of the multiplication of quaternions, we still manage to introduce the concepts of determinant, rank, eigenvalues, and relations of similarity.
- Chapter 10 Studies the concepts of monomials, polynomials and binomials involving quaternion numbers.

It is with great pleasure that we express our appreciation to all those who have expressed support, enthusiasm and encouragement in this adventure. We are forever indebted to our families and close friends for their patience, understanding and support: Lucília and Mário Morais, José António Morais, Ana Beatriz Pistola, Constança Sofia Morais, Francisco Elías, Laura, Borislav, Gerard and Nathalie, and Martina Spröbig. As regards to the present edition, our thanks go to Helmuth Malonek (Aveiro/Portugal), Isabel Cação (Aveiro/Portugal), Klaus Gürlebeck (Weimar/Germany), Tao Qian (Macau/China), Mahmoud Abul-Ez (Sohag/Egypt), Kou Kit Ian (Macau/China), Eckhard Hitzer (Fukui/Japan), Saburo Saitoh (Aveiro/Portugal), Hoai Le (Freiberg/Germany), and Inês Matos (Aveiro/Portugal) for helpful discussions and encouragement, and we especially thank the editor Thomas Hempfling (Birkhäuser) for the meticulous care with which he examined the entire manuscript. We are also grateful to the students who participated in the course “Function theory in higher dimensions” at the Technical University of Mining, Freiberg (Germany), whose enthusiasm, interest and dedication are admirable. For financial aid, the first named author wishes to express his gratitude to the Portuguese Foundation for Science and Technology (“FCT–Fundação para a Ciência e a Tecnologia”) via the postdoctoral grant SFRH/BPD/66342/2009. The first author’s work is also supported by *FEDER* funds through *COMPETE*–Operational Programme Factors of Competitiveness (“Programa Operacional Factores de Competitividade”) and by Portuguese funds through the *Center for Research and Development in Mathematics and Applications* (University of Aveiro) and the FCT, within project PEst-C/MAT/UI4106/2011 with *COMPETE* number FCOMP-01-0124-FEDER-022690.

Although the examples and exercises have been tested several times, we apologize in advance for any errors (typos) or just plain mistakes that you may find, and kindly ask you to bring them to our attention.

Aveiro, Portugal
Sofia, Bulgaria
Freiberg, Germany
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Contents

1	An Introduction to Quaternions	1
1.1	Basic Units	3
1.2	Scalar and Vector Parts	4
1.3	Convention	4
1.4	Equality	5
1.5	Arithmetic Operations	5
1.6	Special Quaternions	7
1.7	Decomposition of Quaternions	8
1.8	Roots	9
1.9	Quaternion Conjugation	9
1.10	Quaternion Modulus and Quaternion Inverse	10
1.11	Quaternion Quotient	11
1.12	Triangle Inequalities	12
1.13	Quaternion Dot Product	13
1.14	Quaternion Cross Product	14
1.15	Mixed Product	16
1.16	Development Formula	17
1.17	Sum Identity for the Double Vector Product	17
1.18	Lagrange Identity	18
1.19	Quaternion Outer and Even Products	19
1.20	Equivalent Quaternions	19
1.21	Polar Form of a Quaternion	22
1.22	Quaternion Sign and Quaternion Argument	23
1.23	Quaternion Argument of a Product	24
1.24	Principal Argument	24
1.25	de Moivre's Formula	24
1.26	Failure for Noninteger Powers	25
1.27	First Matrix Representation of Quaternions	26
1.28	Second Matrix Representation of Quaternions	28
1.29	Advanced Practical Exercises	29
2	Quaternions and Spatial Rotation	35
2.1	Rotations	36
2.2	Composition of Rotations	39

2.3	Rotation Matrix Representation	40
2.4	Quaternion Representation of Rotations	42
2.5	Euler-Rodrigues Formula	46
2.6	Applications to Plane Geometry	47
2.7	Advanced Practical Exercises.....	49
3	Quaternion Sequences	53
3.1	Quaternion Sequence.....	53
3.2	Scalar and Vector Parts of a Quaternion Sequence	54
3.3	Symmetric, Conjugate, Modulus and Inverse of a Quaternion Sequence.....	54
3.4	Arithmetic Operations.....	54
3.5	Convergence of a Quaternion Sequence.....	56
3.6	Divergence of a Quaternion Sequence	56
3.7	Criterion for Convergence	58
3.8	Certain 'Property Eventually'	59
3.9	Quaternion Subsequence	60
3.10	Expression 'of the Order of'	60
3.11	Equivalent Convergence Criteria	61
3.12	Cauchy Quaternion Sequence	61
3.13	Cauchy Convergence Test.....	62
3.14	Summation by Parts	63
3.15	Abel Transformation	63
3.16	Advanced Practical Exercises.....	64
4	Quaternion Series and Infinite Products	69
4.1	Quaternion Series	69
4.2	Arithmetic Properties of Quaternion Series	70
4.3	Geometric Quaternion Series	70
4.4	A Necessary Condition for Convergence	72
4.5	The n th Term Test for Divergence.....	73
4.6	Absolute and Conditional Convergence of a Quaternion Series ...	73
4.7	Root Test.....	74
4.8	Ratio Test	75
4.9	Quaternion Sequences of Bounded Variation	76
4.10	Dirichlet's Test	76
4.11	Alternating Series Test	77
4.12	Abel's Test.....	77
4.13	Quaternion Power Series.....	78
4.14	Radius of Convergence.....	79
4.15	Theoretical Radius	81
4.16	The Arithmetic of Quaternion Power Series	81
4.17	Infinite Products of Quaternion Numbers	82
4.18	Absolute and Conditional Convergence of a Quaternion Product	82
4.19	Advanced Practical Exercises.....	83

5	Exponents and Logarithms	87
5.1	Quaternion Natural Exponential Function	88
5.2	Modulus, Argument, and Conjugate of the Quaternion Exponential Function	91
5.3	Quaternion Natural Logarithm Function	92
5.4	Limits of the Quaternion Exponential and Logarithm Functions	93
5.5	Principal Value of a Quaternion Natural Logarithm Function	94
5.6	The Inverse of the Quaternion Natural Logarithm Function	96
5.7	Quaternion Power Function	97
5.8	General Properties of the Quaternion Power Function	98
5.9	Principal Value of a Quaternion Power Function	100
5.10	Advanced Practical Exercises	100
6	Trigonometric Functions	107
6.1	Quaternion Sine and Cosine Functions	107
6.2	Trigonometric Identities	109
6.3	Trigonometric Equations	110
6.4	Zeros	111
6.5	Quaternion Tangent and Secant Functions	113
6.6	Quaternion Cotangent and Cosecant Functions	114
6.7	Advanced Practical Exercises	115
7	Hyperbolic Functions	117
7.1	Quaternion Sine and Cosine Hyperbolic Functions	117
7.2	Hyperbolic Identities	118
7.3	Equations with Hyperbolic Functions	120
7.4	Relation to Quaternion Sine and Cosine Functions	121
7.5	Zeros	121
7.6	Quaternion Tangent and Secant Functions	122
7.7	Quaternion Cotangent and Cosecant Hyperbolic Functions	123
7.8	Advanced Practical Exercises	123
8	Inverse Hyperbolic and Trigonometric Functions	125
8.1	Quaternion Inverse Hyperbolic Sine and Cosine Functions	125
8.2	Quaternion Inverse Hyperbolic Tangent Function	129
8.3	Quaternion Inverse Trigonometric Sine and Cosine Functions	130
8.4	Quaternion Inverse Trigonometric Tangent Function	130
8.5	Advanced Practical Exercises	131
9	Quaternion Matrices	133
9.1	Quaternion Matrices	133
9.2	Equality	134
9.3	Rank	134
9.4	Matrix Arithmetic and Operations	134
9.5	Special Quaternion Matrices	136
9.6	Basic Notions of Quaternion Matrices	137

9.7	Inverse of a Quaternion Matrix	138
9.8	Quaternion Conjugate of a Matrix	139
9.9	Quaternion Transpose, Conjugate Transpose, and Inverse Matrices	140
9.10	Positive and Semi-positive Definite Quaternion Matrices	141
9.11	Determinant of a Quaternion Matrix	141
9.12	Dieudonné Determinant	143
9.13	Quaternion Eigenvalues and Eigenvectors	144
9.14	Spectrum of a Quaternion Matrix	145
9.15	Right Eigenvalue Equalities	145
9.16	Advanced Practical Exercises	146
10	Monomials, Polynomials and Binomials	149
10.1	Quaternion Monomials	150
10.2	Quaternion Monomials Arithmetic and Operations	151
10.3	Quaternion Polynomials	152
10.4	Factoring and Roots of Quaternion Polynomials	152
10.5	Quaternion Binomial	154
10.6	Basic Relations of Quaternion Binomials	154
10.7	Inverse of the Quaternion Binomial	157
10.8	Basic Relations of Quaternion Inverse Binomials	158
10.9	Advanced Practical Exercises	162
11	Solutions	165
	Bibliography	211
	Index	215