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A Complex Analysis Problem Book

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It is a pleasure to thank the Earl Katz family for endowing the chair (Earl Katz Family Chair in Algebraic System Theory), which supported this research. It is also a pleasure to dedicate this work to my daughter Anaëlle.

Contents

Prologue

Ayons cependant le courage de ressasser des faits connus; ils le sont souvent moins qu'on ne croit.

Marguerite Yourcenar, Sous bénéfice d'inventaire, [149, p. 61].

Prologue

The topic of this book is the theory of complex-valued functions of a complex variable, which are defined on an open subset Ω of the complex plane $\mathbb{C}$ and admit a derivative at every point $z_0 \in \Omega$:

$$\forall z_0 \in \Omega, \quad \exists \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}.$$

Such functions bear various names: They are said to be *holomorphic* in Ω, or *analytic* in Ω; the terms differentiable, $\mathbb{C}$-differentiable and regular are also used. A key result in the theory is the equivalence between $\mathbb{C}$-differentiability in an open set Ω and analyticity of the function, that is, the existence of a power series expansion

$$f(z) = \sum_{n=0}^{\infty} f_n (z - z_0)^n \tag{0.0.1}$$

in a neighborhood of every point $z_0 \in \Omega$. In the first chapters we will make a distinction between the terms *holomorphic* and *analytic*. After the discussion of Cauchy's theorem we will use these terms interchangeably, and mostly use the term *analytic*. We also use mainly the term *analytic* in the discussion below.

The audience we have in mind consists of undergraduate students from mathematics and electrical engineering, with an eye on advanced students from both tracks. Analytic functions are the bread and butter of mathematicians. For engineers, analytic functions appear everywhere, in particular in the theory of linear dynamical systems, signal processing, circuit theory, sampling theorems, optimal control, to name a few. For instance, a motivation for an engineering student would be to know that transfer functions of discrete-time shift-invariant dissipative linear systems are functions analytic in the open unit disk, or in a half-plane, and bounded by one in modulus there (the celebrated Schur functions); see for instance the book [5]. Unfortunately, most, if not all, electrical engineering students do not know what a transfer function is when they begin studying the theory of analytic functions. For the convenience of engineering students we give, in the second part of this prologue, a short discussion of time-invariant linear bounded systems, and their connections to complex variables.

The book consists of four parts. The first two parts, respectively entitled *Complex numbers* and *Function of a complex variable*, form the bulk of the book. Most of the exercises presented in these two parts have been given in the past years by the author in classes on *Introduction to the theory of functions of a complex variable* for second year electrical engineering students, and *Theory of functions of a complex variable* for mathematics students, at the department of mathematics at Ben-Gurion University. The exercises rely only on classical real analysis, but sometimes we use measure theory (mainly via the dominated convergence theorem) to avoid lengthy arguments. Study of some special Hilbert spaces of analytic functions is also scattered in the text, and requires some elementary functional analysis. When studying a function analytic in a domain (for instance in the open unit disk), we will usually (but not always) assume that it is analytic in a neighborhood of this domain, to avoid problems with boundary values. The student will in particular meet in this second part, in simplified forms, Bohr's inequality and the Herglotz integral representation of a function analytic in the open unit disk, and with a real positive part there.

The third part, entitled *Applications and more advanced topics*, contains more advanced material, which was taught by the author to graduate students and also to undergraduate students from the double major program *mathematics and electrical engineering* at Ben-Gurion University. Topics include harmonic functions, conformal mappings and a brief introduction to the theory of linear systems. We hope to come back to these topics, and discuss other advanced topics such as elliptic functions, various aspects of several complex variables and Riemann surfaces in a sequel to the present book.

For the convenience of the reader, we give in the first three parts of the book a number of reminders of known facts from complex analysis, mostly without proofs, in the text. The solutions of most of the exercises are presented at the end of the chapter where they are given.

The fourth part, entitled *Advanced prerequisites*, contains material from real analysis, topology, functional analysis and measure theory, which are needed to solve the exercises (and, in fact, to fully understand a first course on complex variables). Since we mention in the text a number of Hilbert spaces of analytic functions, we also have taken the liberty of mentioning the definition of a reproducing kernel Hilbert space.

In the first weeks of a first course on complex analysis, motivations and applications of the theory are not apparent. Moreover, some results *look like real variable calculus*. One of the difficulties for students who take a complex variable course is that the complex derivative obeys the same rule as the familiar derivative from real analysis. Moreover, the familiar power series of $\sin x, \cos x, ...$ pop up, and it is not clear what the novelty is. After a number of weeks into the course, the student finally sees the proof that a function which is $\mathbb{C}$-differentiable in an open subset of the complex plane admits derivatives of all order, and in fact, admits a power series expansion around every point: The function is *analytic* in the given open set. The student needs to be somewhat patient, to understand slowly the differences between real and complex analysis.

To help the student cope with the difficulties mentioned in the previous paragraph, one approach, sometimes taken by the author, is to skip most of the preliminary material on complex numbers, discuss quickly the notions of continuity and rush to the Cauchy-Riemann equations. One can then already define the exponential function as

$$e^z = e^x(\cos y + i \sin y), \quad z = x + iy \in \mathbb{C},$$

and proceed.

In the present book, we have chosen a slower, and maybe non-standard path for our exposition. We devote the three first chapters of the book to exercises on complex numbers, or complex functions, but without mentioning analyticity. There, the students already meet a variety of functions, such as Blaschke products, the Weierstrass function, and the representation of $\sin z$ as an infinite product. The definition and construction of these functions can be realized without using analyticity. Later in the book, the student will see that these are key examples of analytic functions. Of course, such an approach delays the exercises on analytic functions *per se*, but we think this gives time to the students to absorb at their own pace these difficult examples.

Trying to prove the following formulas using real analysis might provide a student motivation to study complex analysis:

$$\sum_{n=1}^{\infty} r^n \sin(n\theta) = \frac{r \sin\theta}{1 + r^2 - 2r\cos\theta}, \quad r \in [0, 1), \quad \theta \in \mathbb{R},$$

$$z^{2n} + 1 = \prod_{k=0}^{n-1} (z^2 - 2z\cos(\frac{2k+1}{2n}\pi) + 1),$$

$$\int_{\mathbb{R}} \cos t^2 dt = \sqrt{\frac{\pi}{2}},$$

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{7^n} = \sqrt{\frac{7}{3}}.$$

All these formulas are readily proved using complex analysis methods: The first identity is easily proved by a purely real analysis method (multiply both sides by the denominator $1 + r^2 - 2r\cos\theta$, and use a trigonometric identity). The second identity could also, in principle, be directly proved without resorting to complex numbers. See the discussion after the proof of Exercise 1.5.5, where one hints at such a proof using the *completing the square* argument. On the other hand, these two identities have very easy proofs using complex numbers. The proofs of the other two identities use the theory of analytic functions, and in particular Cauchy's theorem (in its weak form), or as in [125, p. 103], the theory of power series, for the integral and the residue theorem for the sum. See respectively Exercises 3.4.9, 1.5.5, 5.2.7 and 7.3.10. The third formula, called Fresnel's integral, can also be computed by real analysis methods. We recall the references at the appropriate place in the text. Still, using complex analysis to compute this integral is, in our opinion, a striking example of the strength of the methods involved.

Linear time-invariant systems

This very short discussion is intended in particular for electrical engineering students, but should be of interest to mathematicians as well. We freely use notions such as measures, positive definite functions, and stochastic processes in the discussion. Some of these notions are recalled later in the book, and we send the reader to the index, to find the exact places where the definitions are given.

A *discrete time system* in engineering is often (but not always!) modeled by an input-output relation

$$(u_n)_{n\in\mathbb{N}_0} \mapsto (y_n)_{n\in\mathbb{N}_0},$$

where $u = (u_n)_{n\in\mathbb{N}_0}$ is the input sequence and $y = (y_n)_{n\in\mathbb{N}_0}$ is the output sequence. One writes this as

$$Tu = y,$$

where T is a possibly non-linear operator between spaces of sequences to be fixed depending on the context. We take in this prologue the spaces of input sequences and output sequences to be both equal to the space ℓ_2 of square summable sequences. Thus

$$\sum_{n=0}^{\infty} |u_n|^2 < \infty \quad \text{and} \quad \sum_{n=0}^{\infty} |y_n|^2 < \infty.$$

These sums can be interpreted as the energy of the signals, and the above inequalities mean that u and y have finite energy.

We assume that the system is:

(1) Linear (that is the operator T is linear from ℓ_2 into itself).

(2) Bounded (that is, T is a bounded, or equivalently, continuous operator).

(3) Time-invariant (we also say shift-invariant): If

$$(u_n)_{n\in\mathbb{N}_0} \mapsto (y_n)_{n\in\mathbb{N}_0},$$

then

$$(u_{n-1})_{n\in\mathbb{N}_0} \mapsto (y_{n-1})_{n\in\mathbb{N}_0},$$

where we set $u_{-1} = y_{-1} = 0$. In other words, if

$$(u_0, u_1, u_2, \ldots) \mapsto (y_0, y_1, y_2, \ldots),$$

then

$$(0, u_0, u_1, u_2, \ldots) \mapsto (0, y_0, y_1, y_2, \ldots).$$

It is proved, using functional analysis tools, that such a system is defined by a convolution operator: There is a sequence $(h_0, h_1, h_2, \ldots)$ such that

$$y_n = \sum_{j=0}^{n} h_{n-j} u_j, \quad n = 0, 1, \ldots. \tag{0.0.2}$$

The z-transform of the sequence $(u_n)_{n\in\mathbb{N}}$ is by definition

$$u(z) = \sum_{n=0}^{\infty} u_n z^n,$$

and is convergent in the open unit disk since $\sum_{n=0}^{\infty} |u_n|^2 < \infty$. The sequence $(h_n)_{n\in\mathbb{N}_0}$ is called the *impulse response* of the system, and its z-transform $h(z) = \sum_{n=0}^{\infty} h_n z^n$ is called its *transfer function*. Taking the z-transform of (0.0.2), we obtain

$$y(z) = h(z)u(z).$$

The fact that the system is bounded (in the sense above) translates into the fact that

$$\sup_{|z|<1} |h(z)| < \infty.$$

The transfer function is a function *analytic and bounded in modulus* in the open unit disk.

This is a first example where analytic functions, blended with appropriate tools from functional analysis not detailed here, appear in electrical engineering. The theory of complex variables allows us to study various problems related to the transfer function (interpolation and approximation for instance), which in counterpart allow us to approximate, or synthesize the system.

The second example we present is related to the theory of continuous time second-order wide sense stationary processes. Such a process $(x(t))_{t\in\mathbb{R}}$ has a covariance function

$$E(x(t)\overline{x(s)}) = r(t - s),$$

where $E(\cdot)$ denotes the expected value, which depends only on the difference $t-s$. Furthermore, the function $r(t-s)$ is positive definite. Since, by the Cauchy-Schwarz inequality,

$$|r(t)| \leq r(0), \tag{0.0.3}$$

the function

$$\varphi(\lambda) = \int_0^\infty e^{i\lambda t} r(t)dt \tag{0.0.4}$$

is well defined in the open upper half-plane $\mathbb{C}_+$. It is analytic and has a positive real part there, as follows from the identity

$$\frac{\varphi(\lambda) + \overline{\varphi(w)}}{-i(\lambda - \overline{w})} = \iint_{[0,\infty)\times[0,\infty)} e^{i\lambda t} e^{-is\overline{w}} r(t-s)dtds, \quad \lambda, w \in \mathbb{C}_+. \tag{0.0.5}$$

The fact that the function φ has a positive real part in $\mathbb{C}_+$ has a number of key consequences. In particular, various interpolation problems, which have applications to the prediction problem for the process, can be solved in an explicit way. Furthermore, the Herglotz representation theorem asserts that one can write

$$\varphi(\lambda) = \frac{1}{i}\int_{\mathbb{R}} \frac{d\mu(t)}{t-\lambda},$$

where $d\mu$ is a probability measure (a more general form of this formula appears later in the book). When $d\mu$ is absolutely continuous with respect to the Lebesgue measure, its derivative is the spectral density of the process[1]. We note that discrete counterparts of formulas (0.0.4) and (0.0.5) are given in the sequel. See (4.4.16) and (4.4.17).

The third example we present here also pertains to the continuous time case: A continuous signal (for instance, the voice) may be modeled by an expression of the form

$$f(t) = \frac{1}{2F}\int_{[-F,F]} e^{-itu} m(u)du, \tag{0.0.6}$$

where the function m, say continuous in this prologue, denotes the spectrum. The representation (0.0.6) expresses that the signal f is built from frequencies in a limited band. The function f is analytic in the complex plane. Its special form (0.0.6) allows us to prove the *sampling theorem*

$$f(t) = \sum_{n\in\mathbb{Z}} f(\frac{\pi n}{F})\frac{\sin(Ft-n\pi)}{Ft-n\pi},$$

where the limit is pointwise (and in a Hilbert space norm too, as explained later in the book). This formula should be a surprise to the students: How can one recover a function of a continuous argument from a discrete number of its values? This

[1] One can, equivalently, also apply Bochner's theorem directly to r to obtain these last results

is possible because of the special properties of f as an analytic function (more precisely, as an entire function of bounded exponential type).

These three examples should at least suggest to the student that the theory of analytic functions has fruitful applications in electrical engineering and signal processing.

Final remarks

The theory of functions of a complex variable is the topic of numerous excellent books, of which we mention [4], [12], [28], [31], [91], [106], to name a few. Classics such as [125] are worth being studied in detail. Interesting sources for exercises are the book of Polya-Szegö [118], the Berkeley entrance exams book [39], and the books of exercises [48] and [115]. Giving precise references to all exercises is a Sisyphean task, and we apologize in advance for any omission.

Finally we conclude with some notation: We use $\mathbb{N} = \{1, 2, \ldots,\}$ for the positive integers, and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The integers are denoted by $\mathbb{Z}$, and $\mathbb{D}$ denotes the open unit disk. The unit circle is denoted by $\mathbb{T}$, and $\mathbb{C}_r$ stands for the open right half-plane. The open upper half-plane is denoted by $\mathbb{C}_+$.

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