Graduate Texts in Mathematics 282

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Sheldon Axler

Measure, Integration & Real Analysis

Springer Open

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ISSN 0072-5285 ISSN 2197-5612 (electronic)
Graduate Texts in Mathematics
ISBN 978-3-030-33142-9 ISBN 978-3-030-33143-6 (eBook)
https://doi.org/10.1007/978-3-030-33143-6

Mathematics Subject Classification (2010): 28A, 42A, 46B, 46C, 47A, 60A

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The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland

Dedicated to

Paul Halmos, Don Sarason, and Allen Shields,

the three mathematicians who most
helped me become a mathematician.

About the Author

Sheldon Axler was valedictorian of his high school in Miami, Florida. He received his AB from Princeton University with highest honors, followed by a PhD in Mathematics from the University of California at Berkeley.

As a postdoctoral Moore Instructor at MIT, Axler received a university-wide teaching award. He was then an assistant professor, associate professor, and professor at Michigan State University, where he received the first J. Sutherland Frame Teaching Award and the Distinguished Faculty Award.

Axler received the Lester R. Ford Award for expository writing from the Mathematical Association of America in 1996. In addition to publishing numerous research papers, he is the author of six mathematics textbooks, ranging from freshman to graduate level. His book *Linear Algebra Done Right* has been adopted as a textbook at over 300 universities and colleges.

Axler has served as Editor-in-Chief of the *Mathematical Intelligencer* and Associate Editor of the *American Mathematical Monthly*. He has been a member of the Council of the American Mathematical Society and a member of the Board of Trustees of the Mathematical Sciences Research Institute. He has also served on the editorial board of Springer's series Undergraduate Texts in Mathematics, Graduate Texts in Mathematics, Universitext, and Springer Monographs in Mathematics.

He has been honored by appointments as a Fellow of the American Mathematical Society and as a Senior Fellow of the California Council on Science and Technology.

Axler joined San Francisco State University as Chair of the Mathematics Department in 1997. In 2002, he became Dean of the College of Science & Engineering at San Francisco State University. After serving as Dean for thirteen years, he returned to a regular faculty appointment as a professor in the Mathematics Department.

Cover figure: Hölder's Inequality, which is proved in Section 7A.

Contents

Preface for Students

You are about to immerse yourself in serious mathematics, with an emphasis on attaining a deep understanding of the definitions, theorems, and proofs related to measure, integration, and real analysis. This book aims to guide you to the wonders of this subject.

You cannot read mathematics the way you read a novel. If you zip through a page in less than an hour, you are probably going too fast. When you encounter the phrase *as you should verify*, you should indeed do the verification, which will usually require some writing on your part. When steps are left out, you need to supply the missing pieces. You should ponder and internalize each definition. For each theorem, you should seek examples to show why each hypothesis is necessary.

Working on the exercises should be your main mode of learning after you have read a section. Discussions and joint work with other students may be especially effective. Active learning promotes long-term understanding much better than passive learning. Thus you will benefit considerably from struggling with an exercise and eventually coming up with a solution, perhaps working with other students. Finding and reading a solution on the internet will likely lead to little learning.

As a visual aid, throughout this book definitions are in yellow boxes and theorems are in blue boxes, in both print and electronic versions. Each theorem has an informal descriptive name. The electronic version of this manuscript has links in blue.

Please check the website below (or the Springer website) for additional information about the book. These websites link to the electronic version of this book, which is free to the world because this book has been published under Springer's Open Access program. Your suggestions for improvements and corrections for a future edition are most welcome (send to the email address below).

The prerequisite for using this book includes a good understanding of elementary undergraduate real analysis. You can download from the website below or from the Springer website the document titled *Supplement for Measure, Integration & Real Analysis*. That supplement can serve as a review of the elementary undergraduate real analysis used in this book.

Best wishes for success and enjoyment in learning measure, integration, and real analysis!

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Preface for Instructors

You are about to teach a course, or possibly a two-semester sequence of courses, on measure, integration, and real analysis. In this textbook, I have tried to use a gentle approach to serious mathematics, with an emphasis on students attaining a deep understanding. Thus new material often appears in a comfortable context instead of the most general setting. For example, the Fourier transform in Chapter 11 is introduced in the setting of $\mathbf{R}$ rather than $\mathbf{R}^n$ so that students can focus on the main ideas without the clutter of the extra bookkeeping needed for working in $\mathbf{R}^n$.

The basic prerequisite for your students to use this textbook is a good understanding of elementary undergraduate real analysis. Your students can download from the book's website (http://measure.axler.net) or from the Springer website the document titled *Supplement for Measure, Integration & Real Analysis*. That supplement can serve as a review of the elementary undergraduate real analysis used in this book.

As a visual aid, throughout this book definitions are in yellow boxes and theorems are in blue boxes, in both print and electronic versions. Each theorem has an informal descriptive name. The electronic version of this manuscript has links in blue.

Mathematics can be learned only by doing. Fortunately, real analysis has many good homework exercises. When teaching this course, during each class I usually assign as homework several of the exercises, due the next class. I grade only one exercise per homework set, but the students do not know ahead of time which one. I encourage my students to work together on the homework or to come to me for help. However, I tell them that getting solutions from the internet is not allowed and would be counterproductive for their learning goals.

If you go at a leisurely pace, then covering Chapters 1–5 in the first semester may be a good goal. If you go a bit faster, then covering Chapters 1–6 in the first semester may be more appropriate. For a second-semester course, covering some subset of Chapters 6 through 12 should produce a good course. Most instructors will not have time to cover all those chapters in a second semester; thus some choices need to be made. The following chapter-by-chapter summary of the highlights of the book should help you decide what to cover and in what order:

- **Chapter 1:** This short chapter begins with a brief review of Riemann integration. Then a discussion of the deficiencies of the Riemann integral helps motivate the need for a better theory of integration.

- **Chapter 2:** This chapter begins by defining outer measure on $\mathbf{R}$ as a natural extension of the length function on intervals. After verifying some nice properties of outer measure, we see that it is not additive. This observation leads to restricting our attention to the σ-algebra of Borel sets, defined as the smallest σ-algebra on $\mathbf{R}$ containing all the open sets. This path leads us to measures.

After dealing with the properties of general measures, we come back to the setting of **R**, showing that outer measure restricted to the σ-algebra of Borel sets is countably additive and thus is a measure. Then a subset of **R** is defined to be Lebesgue measurable if it differs from a Borel set by a set of outer measure 0. This definition makes Lebesgue measurable sets seem more natural to students than the other competing equivalent definitions. The Cantor set and the Cantor function then stretch students' intuition.

Egorov's Theorem, which states that pointwise convergence of a sequence of measurable functions is close to uniform convergence, has multiple applications in later chapters. Luzin's Theorem, back in the context of **R**, sounds spectacular but has no other uses in this book and thus can be skipped if you are pressed for time.

- **Chapter 3:** Integration with respect to a measure is defined in this chapter in a natural fashion first for nonnegative measurable functions, and then for real-valued measurable functions. The Monotone Convergence Theorem and the Dominated Convergence Theorem are the big results in this chapter that allow us to interchange integrals and limits under appropriate conditions.

- **Chapter 4:** The highlight of this chapter is the Lebesgue Differentiation Theorem, which allows us to differentiate an integral. The main tool used to prove this result cleanly is the Hardy–Littlewood maximal inequality, which is interesting and important in its own right. This chapter also includes the Lebesgue Density Theorem, showing that a Lebesgue measurable subset of **R** has density 1 at almost every number in the set and density 0 at almost every number not in the set.

- **Chapter 5:** This chapter deals with product measures. The most important results here are Tonelli's Theorem and Fubini's Theorem, which allow us to evaluate integrals with respect to product measures as iterated integrals and allow us to change the order of integration under appropriate conditions. As an application of product measures, we get Lebesgue measure on $\mathbf{R}^n$ from Lebesgue measure on **R**. To give students practice with using these concepts, this chapter finds a formula for the volume of the unit ball in $\mathbf{R}^n$. The chapter closes by using Fubini's Theorem to give a simple proof that a mixed partial derivative with sufficient continuity does not depend upon the order of differentiation.

- **Chapter 6:** After a quick review of metric spaces and vector spaces, this chapter defines normed vector spaces. The big result here is the Hahn–Banach Theorem about extending bounded linear functionals from a subspace to the whole space. Then this chapter introduces Banach spaces. We see that completeness plays a major role in the key theorems: Open Mapping Theorem, Inverse Mapping Theorem, Closed Graph Theorem, and Principle of Uniform Boundedness.

- **Chapter 7:** This chapter introduces the important class of Banach spaces $L^p(\mu)$, where $1 \le p \le \infty$ and μ is a measure, giving students additional opportunities to use results from earlier chapters about measure and integration theory. The crucial results called Hölder's inequality and Minkowski's inequality are key tools here. This chapter also shows that the dual of ℓ^p is $\ell^{p'}$ for $1 \le p < \infty$.

Chapters 1 through 7 should be covered in order, before any of the later chapters. After Chapter 7, you can cover Chapter 8 or Chapter 12.

- **Chapter 8:** This chapter focuses on Hilbert spaces, which play a central role in modern mathematics. After proving the Cauchy–Schwarz inequality and the Riesz Representation Theorem that describes the bounded linear functionals on a Hilbert space, this chapter deals with orthonormal bases. Key results here include Bessel's inequality, Parseval's identity, and the Gram–Schmidt process.

- **Chapter 9:** Only positive measures have been discussed in the book up until this chapter. In this chapter, real and complex measures get consideration. These concepts lead to the Banach space of measures, with total variation as the norm. Key results that help describe real and complex measures are the Hahn Decomposition Theorem, the Jordan Decomposition Theorem, and the Lebesgue Decomposition Theorem. The Radon–Nikodym Theorem is proved using von Neumann's slick Hilbert space trick. Then the Radon–Nikodym Theorem is used to prove that the dual of $L^p(\mu)$ can be identified with $L^{p'}(\mu)$ for $1 < p < \infty$ and μ a (positive) measure, completing a project that started in Chapter 7.

  The material in Chapter 9 is not used later in the book. Thus this chapter can be skipped or covered after one of the later chapters.

- **Chapter 10:** This chapter begins by discussing the adjoint of a bounded linear map between Hilbert spaces. Then the rest of the chapter presents key results about bounded linear operators from a Hilbert space to itself. The proof that each bounded operator on a complex nonzero Hilbert space has a nonempty spectrum requires a tiny bit of knowledge about analytic functions. Properties of special classes of operators (self-adjoint operators, normal operators, isometries, and unitary operators) are described.

  Then this chapter delves deeper into compact operators, proving the Fredholm Alternative. The chapter concludes with two major results: the Spectral Theorem for compact operators and the popular Singular Value Decomposition for compact operators. Throughout this chapter, the Volterra operator is used as an example to illustrate the main results.

  Some instructors may prefer to cover Chapter 10 immediately after Chapter 8, because both chapters live in the context of Hilbert space. I chose the current order to give students a breather between the two Hilbert space chapters, thinking that being away from Hilbert space for a little while and then coming back to it might strengthen students' understanding and provide some variety. However, covering the two Hilbert space chapters consecutively would also work fine.

- **Chapter 11:** Fourier analysis is a huge subject with a two-hundred year history. This chapter gives a gentle but modern introduction to Fourier series and the Fourier transform.

  This chapter first develops results in the context of Fourier series, but then comes back later and develops parallel concepts in the context of the Fourier transform. For example, the Fourier coefficient version of the Riemann–Lebesgue Lemma is proved early in the chapter, with the Fourier transform version proved later in the chapter. Other examples include the Poisson kernel, convolution, and the Dirichlet problem, all of which are first covered in the context of the unit disk and unit circle; then these topics are revisited later in the context of the half-plane and real line.

Convergence of Fourier series is proved in the L^2 norm and also (for sufficiently smooth functions) pointwise. The book emphasizes getting students to work with the main ideas rather than on proving all possible results (for example, pointwise convergence of Fourier series is proved only for twice continuously differentiable functions rather than using a weaker hypothesis).

The proof of the Fourier Inversion Formula is the highlight of the material on the Fourier transform. The Fourier Inversion Formula is then used to show that the Fourier transform extends to a unitary operator on $L^2(\mathbf{R})$.

This chapter uses some basic results about Hilbert spaces, so it should not be covered before Chapter 8. However, if you are willing to skip or hand-wave through one result that helps describe the Fourier transform as an operator on $L^2(\mathbf{R})$ (see 11.87), then you could cover this chapter without doing Chapter 10.

- **Chapter 12:** A thorough coverage of probability theory would require a whole book instead of a single chapter. This chapter takes advantage of the book's earlier development of measure theory to present the basic language and emphasis of probability theory. For students not pursuing further studies in probability theory, this chapter gives them a good taste of the subject. Students who go on to learn more probability theory should benefit from the head start provided by this chapter and the background of measure theory.

  Features that distinguish probability theory from measure theory include the notions of independent events and independent random variables. In addition to those concepts, this chapter discusses standard deviation, conditional probabilities, Bayes' Theorem, and distribution functions. The chapter concludes with a proof of the Weak Law of Large Numbers for independent identically distributed random variables.

  You could cover this chapter anytime after Chapter 7.

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I enjoy keeping track of where my books are used as textbooks. If you use this book as the textbook for a course, please let me know.

Best wishes for teaching a successful class on measure, integration, and real analysis!

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Acknowledgments

I owe a huge intellectual debt to the many mathematicians who created real analysis over the past several centuries. The results in this book belong to the common heritage of mathematics. A special case of a theorem may first have been proved by one mathematician and then sharpened and improved by many other mathematicians. Bestowing accurate credit for all the contributions would be a difficult task that I have not undertaken. In no case should the reader assume that any theorem presented here represents my original contribution. However, in writing this book I tried to think about the best way to present real analysis and to prove its theorems, without regard to the standard methods and proofs used in most textbooks.

The manuscript for this book received an unusually large amount of class testing at several universities before publication. Thus I received many valuable suggestions for improvements and corrections. I am deeply grateful to all the faculty and students who helped class test the manuscript. I implemented suggestions or corrections from the following faculty and students, all of whom helped make this a better book:

> Sunayan Acharya, Ali Al Setri, Nick Anderson, Kevin Bui, Tony Cairatt, Eric Carmody, Timmy Chan, Logan Clark, Sam Coskey, Yerbolat Dauletyarov, Evelyn Easdale, Ben French, Loukas Grafakos, Michael Hanson, Michah Hawkins, Eric Hayashi, Nelson Huang, Jasim Ismaeel, Brody Johnson, Hannah Knight, Oliver Knitter, Chun-Kit Lai, Lee Larson, Vens Lee, Hua Lin, David Livnat, Shi Hao Looi, Dante Luber, Stephanie Magallanes, Jan Mandel, Juan Manfredi, Zack Mayfield, Calib Nastasi, Lamson Nguyen, Kiyoshi Okada, Célio Passos, Isabel Perez, Ricardo Pires, Hal Prince, Noah Rhee, Ken Ribet, Spenser Rook, Arnab Dey Sarkar, Wayne Small, Emily Smith, Keith Taylor, Ignacio Uriarte-Tuero, Alexander Wittmond, Run Yan, Edward Zeng.

Loretta Bartolini, Mathematics Editor at Springer, is owed huge thanks for her multiple major contributions to this project. Paula Francis, the copy editor for this book, provided numerous useful suggestions and corrections that improved the book.

Thanks also to the people who allowed photographs they produced to enhance this book. For a complete list, see the Photo Credits pages near the end of the book.

Special thanks to my wonderful partner Carrie Heeter, whose understanding and encouragement enabled me to work intensely on this book. Our cat Moon, whose picture is on page 44, helped provide relaxing breaks.

Sheldon Axler