Simplicial Methods for Higher Categories

Algebra and Applications

Volume 26

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Segal-type Models of Weak n-Categories

Springer

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ISSN 1572-5553 ISSN 2192-2950 (electronic)
Algebra and Applications
ISBN 978-3-030-05673-5 ISBN 978-3-030-05674-2 (eBook)
https://doi.org/10.1007/978-3-030-05674-2

Mathematics Subject Classification (2010): 18-XX, 55P-XX

This Springer imprint is published by the registered company Springer Nature Switzerland AG.
The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland

To my parents

Preface

The theory of higher categories is a very active area of research which has penetrated diverse fields of science.

Historically the subject was motivated by questions in algebraic topology and mathematical physics, two areas where the most important applications are currently found. Algebraic geometry also makes use of higher categorical notions. More recently higher categories have found their way into logic and computer science, and are also starting to appear in algebra and representation theory. Higher categories can sometimes be used as a common language to describe complex phenomena occurring in these areas.

A plethora of different approaches to higher categories have been developed over the years. Each one represents certain relevant aspects of the abstract notion being modelled, often with a view to supporting a particular ecosystem of applications. At this stage no single approach suits all such contexts, and indeed one might doubt the viability of a universally applicable model. It appears instead that the most prudent approach is to continue the development of all of these important strands, relating them where necessary by explicit comparisons.

The purpose of this monograph is to introduce a new approach to working with higher categories: this is based on a simple higher categorical structure consisting of iterated internal categories (also called n-fold categories) as well as on a new paradigm to weaker higher categorical structures, which is the idea of weak globularity.

We show that our new model, called weakly globular n-fold categories, is suitably equivalent to a model of higher categories that has been studied in great depth, the one introduced by Tamsamani [126] and further studied by Simpson [119].

We achieve this comparison by developing a larger context of 'Segal-type models of weak n-categories', based on multi-simplicial structures, of which both the Tamsamani model and weakly globular n-fold categories are special cases.

The use of simplicial structures to capture higher coherence phenomena has a long history in algebraic topology, starting with the work of Graeme Segal [117], and then the study of categories enriched in simplicial sets by Dwyer and Kan

[48–50], Dwyer et al. [52] and others. More recently, simplicial techniques have underpinned the development of so-called (∞, n)-categories, and several models have been developed and studied by Bergner [23], Bergner and Rezk [27], Barwick and Kan [9], Lurie [89], Joyal [74], Rezk [109, 110] and others.

Simplicial models of (∞, ∞)-categories have been developed by Verity [137], building upon insights from the study of simplicial nerves of strict n-categories initiated by Street [123].

In this work we concentrate on higher structures in the 'truncated' case, where there are higher morphisms only in dimensions 0 up to n. This is intimately connected to the Postnikov tower in algebraic topology. In fact, the algebraic modelling of the building blocks of spaces, the n-types, which are Postnikov sections of spaces, is related to models of weak n-categories via the so-called 'homotopy hypothesis': a good model of weak n-categories should give an algebraic model of n-types in the weak n-groupoid case. We show that our models do satisfy the homotopy hypothesis.

There are long-standing open questions about weak n-categories, both within category theory and in its applications to homotopy theory: for instance the comparison between the simplicial and higher operadic models of higher categories and the algebraic description of the k-invariants of spaces and simplicial categories.

The present work provides a platform where these and other open questions can be studied, as we outline in the last chapter. These questions however go beyond the scope of this work, whose goal is to lay the foundations of this theory.

The potential of our model to tackle these open questions comes from one of the main novelties of our approach: the use of an entirely rigid structure, namely a subcategory of n-fold categories, to model weak n-categories.

The terminology 'rigid structure' refers to the fact that n-fold categories, being iterated internal categories, have associative and unital compositions in n different simplicial directions. In this sense, n-fold categories are a strict higher categorical structure, though they are not the same as strict n-categories, since the higher morphisms in dimensions 0 to up n do not form just a set. In our model, the higher morphisms in dimension k (for each $0 \leq k \leq n-2$) have themselves an $(n-1+k)$-fold categorical structure of a special type which is suitably equivalent to a discrete structure (that is, a set): we call this property the 'weak globularity condition'.

n-Fold structures were used in homotopy theory by Loday [85] for the modelling of connected $(n+1)$-types via cat^n-groups. The idea of weak globularity was first introduced by the author in [102] in an internal setting for the category of cat^n-groups: weakly globular cat^n-groups were shown in [102] to be algebraic models of connected $(n+1)$-types; weak globularity was extended and further studied by Blanc and the author in [29] in the context of general n-types, for which an analogue of Loday's model was not available.

However, none of these works captured the general categorical case. This case necessitates many novel ideas and techniques, such as the use of pseudo-functors to model higher structures and the construction of a rigidification functor from the Tamsamani model to weakly globular n-fold categories.

This work uses a blend of techniques from category theory and simplicial homotopy theory, reviewed in Part I. We therefore hope it will be accessible both to category theorists and to algebraic topologists.

This book is organized into four parts:

Part I Higher Categories: Introduction and Background.
This Part aims to provide the reader with a guide to the rest of the book. It contains a broad introduction to higher categories, some historical development of the notion of weak globularity and a non-technical overview of the main ideas and results we shall encounter. It also covers the main techniques that will be used from category theory and simplicial homotopy theory.

Part II The Three Segal-type Models and Segalic Pseudo-Functors.
In this Part we introduce the three Segal-type models, that is, the categories $\mathsf{Ta}^n_{\mathsf{wg}}$ of weakly globular Tamsamani n-categories and its subcategories Ta^n (Tamsamani n-categories) and $\mathsf{Cat}^n_{\mathsf{wg}}$ (weakly globular n-fold categories). We establish the relation between the category $\mathsf{Cat}^n_{\mathsf{wg}}$ and a class of pseudo-functors which we call Segalic pseudo-functors.

Part III Rigidification of Weakly Globular Tamsamani n-Categories.
The main goal of this Part is the construction of the rigidification functor from weakly globular Tamsamani n-categories to weakly globular n-fold categories.

Part IV Weakly Globular n-Fold Categories as a Model of Weak n-Categories.
This Part contains the construction of the discretization functor from weakly globular n-fold categories to Tamsamani n-categories, and the final results: the equivalence after localization of $\mathsf{Cat}^n_{\mathsf{wg}}$ and Ta^n, exhibiting $\mathsf{Cat}^n_{\mathsf{wg}}$ as a model of weak n-categories, and the proof of the homotopy hypothesis. The last chapter of this Part contains an outline of further directions of applications and of open questions arising from this work.

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Acknowledgements

This work was supported by the EU International Reintegration Grant HOMALGHIGH No 256341, of which I was principal investigator. It also received financial support from the University of Leicester, where I have worked since 2011 and which supported my study leave in 2015; from the Centre of Australian Category Theory at Macquarie University, which hosted me during August–December 2015; and from the University of Chicago, where I visited in 2016.

This work was presented in several talks, where I could gather invaluable feedback from colleagues. In particular I thank the Centre of Australian Category Theory for the opportunity to give a long series of talks at the Australian Category Seminar, and I am particularly indebted for useful feedback from its members, especially Michael Batanin, Steve Lack, Richard Garner, Ross Street, Mark Weber, and Dominic Verity.

I would also like to thank Peter May for giving me the chance to give a series of talks at the University of Chicago and for many useful comments.

I am grateful for the opportunity to present this material at international conferences and at seminars and for the discussions with experts in the field that followed. I thank in particular Joachim Kock, Martin Hyland, and Dorette Pronk for interesting discussions and feedback. I also thank David Blanc, Frank Neumann, Alexander Kurz and Dominic Verity for reading the introductory part of the manuscript and for their helpful comments, and Nicola Gambino for some bibliographical suggestions.

I thank my colleagues in the Department of Mathematics of the University of Leicester for their ongoing support and encouragement.

I express my deep gratitude to the four reviewers of this book, for their careful reading and numerous helpful comments and suggestions.

I finally thank my parents for their invaluable moral support throughout my mathematical life.

Contents

List of Symbols

$[\mathscr{A}, \mathscr{B}]$	Category of functors from $\mathscr{A}$ to $\mathscr{B}$ and natural transformations, Sect. 1.2.4
$A[f]$	Internal equivalence relation corresponding to $f : A \to B$, Definition 5.1.8
$\widetilde{B}$	Functor $\mathsf{GCat}^{\mathsf{n}}_{\mathsf{wg}}/\sim \to$ Ho(n-types), Remark 12.4.5
B	Classifying space. Definition 5.3.1, Sects. 12.3, 12.4
$\mathsf{Cat}^{\mathsf{n}}_{\mathsf{hd}}$	Category of homotopically discrete n-fold categories, Definition 5.1.2
$\mathsf{Cat}\,\mathscr{C}$	Category of internal categories in $\mathscr{C}$, Definition 2.3.1
$\mathsf{Cat}^{\mathsf{n}}(\mathscr{C})$	Category of n-fold internal categories in $\mathscr{C}$, Sect. 2.3
$\mathsf{Cat}^{\mathsf{n}}$	Category of n-fold categories, Sect. 2.5
$\mathsf{Cat}^{\mathsf{n}}_{\mathsf{wg}}$	Category of weakly globular n-fold categories, Definition 6.3.3
Cube(n, t)	Set of (n, t)-hypercubes, Definition 7.2.18
Dec, Dec$'$	Functors décalage, Sect. 2.7
Δ	Simplicial category, Sect. 2.1.2
$\Delta^{n^{op}}$	Product of n copies of Δ^{op}, Sect. 2.1.2
$[\Delta^{op}, \mathscr{C}]$	Category of simplicial objects in $\mathscr{C}$, Sect. 2.1.1
$[\Delta^{n^{op}}, \mathscr{C}]$	Category of n-fold simplicial objects in $\mathscr{C}$, Sect. 2.1.2
$[\Delta^{n^{op}}, \mathsf{Set}]$	Category of n-fold simplicial sets, Sect. 2.2
$d^{(n)}$	Functor $\mathsf{Cat}^{n-1}(\mathscr{C}) \to \mathsf{Cat}^{\mathsf{n}}(\mathscr{C})$, Definition 2.4.11
d_n	Functor $[\Delta^{n-1^{op}}, \mathscr{C}] \to [\Delta^{n^{op}}, \mathscr{C}]$, Definition 2.4.11
$Diag_n$	Multi-diagonal functor, Definition 5.3.1
$Disc_n$	Discretization functor, Definition 12.2.1
D_n	Functor $\mathsf{FCat}^{\mathsf{n}}_{\mathsf{wg}} \to \mathsf{Ta}^{\mathsf{n}}$, Proposition 12.1.4
F_n	Functor $\mathsf{Cat}^{\mathsf{n}}_{\mathsf{wg}} \to \mathsf{Cat}^{\mathsf{n}}_{\mathsf{wg}}$, Proposition 11.2.5
$\overline{F}$	Functor $[\mathscr{I}, \mathscr{C}] \to [\mathscr{I}, \mathscr{D}]$ for $F : \mathscr{C} \to \mathscr{D}$, Definition 2.1.1
$f_n(X)$	Map $F_n X \to X$, Proposition 11.2.5
$\mathsf{FCat}^{\mathsf{n}}_{\mathsf{wg}}$	Definition 11.3.1

$\mathsf{GCat}^{\mathsf{n}}_{\mathsf{wg}}$	Category of groupoidal weakly globular n-fold categories, Definition 12.3.6
$g_n(X)$	Map $G_n X \to X$, Theorem 11.3.6
G_n	Functor $\mathsf{Cat}^{\mathsf{n}}_{\mathsf{wg}} \to \mathsf{FCat}^{\mathsf{n}}_{\mathsf{wg}}$, Theorem 11.3.6
$\mathscr{G}_n$	Fundamental groupoidal weakly globular n-fold category functor, Sect. 12.4, Eq. (12.26)
$\mathsf{Gpd}\,\mathscr{C}$	Category of internal groupoids in $\mathscr{C}$, Definition 2.3.3
Gpd^n	Category of n-fold groupoids, Sect. 2.5
$\mathsf{Gpd}^{\mathsf{n}}_{\mathsf{wg}}$	Category of weakly globular n-fold groupoids, Sect. 12.4
$\mathsf{GSeg}_{\mathsf{n}}$	Groupoidal Segal-type model, Sect. 3.4.3
$\mathsf{GTa}^{\mathsf{n}}$	Category of groupoidal Tamsamani n-categories, Definition 12.3.6
$\mathsf{GTa}^{\mathsf{n}}_{\mathsf{wg}}$	Category of groupoidal weakly globular Tamsamani n-categories, Definition 12.3.1
$\mathscr{H}_n$	Alternative fundamental groupoidal weakly globular n-fold category functor, Definition 12.4.2
$h_n(X)$	Map $V_n(X) \to F_n(X)$, $X \in \mathsf{Cat}^{\mathsf{n}}_{\mathsf{hd}}$, Proof of Proposition 11.2.3
$\underline{k}(1,i)$, $\underline{k}(0,i)$	Remark 8.1.3
Ho(n-types)	Homotopy category of n-types, Sect. 1.3
J_n	Functor n-$\mathsf{Cat} \to [\Delta^{n-1^{op}}, \mathsf{Cat}\,]$, Definition 2.6.3
$\mathsf{LTa}^{\mathsf{n}}_{\mathsf{wg}}$	Definition 9.1.1
n-Cat	Category of strict n-categories, Definition 2.6.1
n-Gpd	Category of strict n-groupoids, Sect. 2.6
N	Nerve functor $N : \mathsf{Cat}\,\mathscr{C} \to [\Delta^{op}, \mathscr{C}]$, Sect. 2.4
$N^{(k)}$	Nerve functor in the kth direction, Definition 2.4.8
$N_{(n)}$	Multinerve functor, Definition 2.4.3
$\mathrm{Or}_{(2)}$	Functor $[\Delta^{op}, \mathsf{Set}] \to [\Delta^{2^{op}}, \mathsf{Set}]$, Fig. 12.4
$\mathrm{Or}_{(3)}$	Functor $[\Delta^{op}, \mathsf{Set}] \to [\Delta^{3^{op}}, \mathsf{Set}]$, Fig. 12.6
$\mathrm{Or}_{(n)}$	Functor $[\Delta^{op}, \mathsf{Set}] \to [\Delta^{n^{op}}, \mathsf{Set}]$, Sect. 12.4
or_n	Ordinal sum $\Delta^n \to \Delta$, Sect. 12.4
p	Functor $[\Delta^{n^{op}}, \mathsf{Set}] \to \mathsf{Set}$, Notational Convention 2.2.4
p_n	Functor $[\Delta^{n^{op}}, \mathsf{Set}] \to \mathsf{Set}$, Definition 2.2.3
$p^{(r)}$	Functor $\mathsf{Seg}_n \to \mathsf{Seg}_r$, Definition 6.3.3, Definition 6.1.8, Definition 5.1.2
$p^{(n)}$	Functor $\mathsf{SegPs}[\Delta^{n^{op}}, \mathsf{Cat}\,] \to \mathsf{Cat}^{\mathsf{n}}_{\mathsf{wg}}$, Definition 8.1.2
P_n	Functor $P_n : \mathsf{Ta}^{\mathsf{n}}_{\mathsf{wg}} \to \mathsf{LTa}^{\mathsf{n}}_{\mathsf{wg}}$, Proof of Theorem 10.2.1
$\mathscr{P}$	Homotopy category functor $[\Delta^{op}, \mathsf{Set}] \to \mathsf{Cat}$, Definition 2.2.2
$\mathscr{P}_n$	Left adjoint to the n-fold nerve, Sect. 12.4
$\mathsf{Ps}[\Delta^{n^{op}}, \mathsf{Cat}\,]$	Category of pseudo-functors, Sect. 4.2
q	Connected components functor, Notational Convention 2.2.4
q_n	Functor $[\Delta^{n^{op}}, \mathsf{Set}] \to \mathsf{Set}$, Definition 2.2.3
$q^{(r)}$	Functor $\mathsf{Seg}_n \to \mathsf{Seg}_r$, Proposition 7.1.7, Corollary 7.1.8

Q_n	Rigidification functor, Theorem 10.2.1
R_0	Functor $\mathsf{FCat}^{\mathsf{n}}_{\mathsf{wg}} \to [\Delta^{op}, \mathsf{FCat}^{\mathsf{n-1}}_{\mathsf{wg}}]$, Definition 12.1.1
$\mathscr{R}_n$	Composite $\mathscr{P}_n \,\mathrm{Or}_{(n)}$, Sect. 12.4.2
$\mathscr{S}$	Singular functor $\mathsf{Top} \to [\Delta^{op}, \mathsf{Set}]$, Sect. 12.4
Seg_n	Segal-type model, Sect. 3.4.2
$\mathsf{SegPs}[\Delta^{n^{op}}, \mathsf{Cat}]$	Category of Segalic pseudo-functors, Definition 8.1.2
$s_n(X)$	Map $Q_n X \to X$, Theorem 10.2.1
St	Strictification functor, Sect. 4.2
Ta^{n}	Category of Tamsamani n-categories, Definition 6.2.1
$\mathsf{Ta}^{\mathsf{n}}_{\mathsf{wg}}$	Category of weakly globular Tamsamani n-categories, Definition 6.1.8
$t_n(X)$	Pseudo-natural transformation $Tr_n X \to X$, Theorem 10.1.1
$\mathscr{T}_n$	Fundamental Tamsamani n-groupoid functor, Theorem 12.3.11
Tr_n	Functor $\mathsf{LTa}^{\mathsf{n}}_{\mathsf{wg}} \to \mathsf{SegPs}[\Delta^{n-1^{op}}, \mathsf{Cat}]$, Theorem 10.1.1
u_X	Map $\mathrm{Dec}\, X \to X$, Sect. 2.7
$v_n(X)$	Map $V_n X \to X$, Proposition 11.2.3
$v_k^{\{r\}}$	Map $X_k^{\{r\}} \to X_1^{\{r\}} \times_{p^{(n-2)} X_0^{\{r\}}} \overset{k}{\cdots} \times_{p^{(n-2)} X_0^{\{r\}}} X_1^{\{r\}}$, Sect. 9.1.2
V_n	Functor $\mathsf{Cat}^{\mathsf{n}}_{\mathsf{hd}} \to \mathsf{Cat}^{\mathsf{n}}_{\mathsf{hd}}$, Proposition 11.2.3
$V(X)$	Map $X(f_0) \to X$, Lemma 11.1.1, Proposition 11.1.5
$w_n(X)$	Map $P_n X \to X$, Proof of Theorem 10.2.1
$X(a,b)$	Hom-$(n-1)$-category of $X \in \mathsf{Seg}_n$, Notation 5.2.4, Definition 6.3.3, Definition 6.1.8
$X(f_0)$	Construction on $X \in \mathsf{Cat}\,\mathscr{C}$ and $f_0 : X'_0 \to X_0$, Lemma 11.1.1
X^d	Discretization of $X \in \mathsf{Cat}^{\mathsf{n}}_{\mathsf{hd}}$, Definition 5.1.2
X^{α}, $X^{\{r\}}$	Definition 2.1.5, Notational Convention 2.5.4
$\gamma_{(n)}$	Discretization map $X \to X^d$ for $X \in \mathsf{Cat}^{\mathsf{n}}_{\mathsf{hd}}$, Definition 5.1.4
$\gamma^{(r)}$	Map $X \to q^{(r)} X$, Lemma 2.2.7, Remark 7.1.9, Remark 7.2.13
μ_k	kth Segal map, Definition 2.1.2
$\hat{\mu}_k$	kth induced Segal map, Definition 2.1.3
$\widetilde{\xi}_i$	Map $\mathsf{Cat}^{\mathsf{n}}(\mathscr{C}) \to \mathsf{Cat}\,(\mathsf{Cat}^{n-1}(\mathscr{C}))$, Proposition 2.4.6
ξ_i	Map $[\Delta^{n^{op}}, \mathscr{C}] \to [\Delta^{op}, [\Delta^{n-1^{op}}, \mathscr{C}]]$, Lemma 2.1.7
Σ_n	Symmetric group, Definition 2.1.5
$\hat{\pi}_1^{(i)}$	Fundamental groupoid functor in direction i, Theorem 12.4.3
$\sim^n$	n-equivalences in Seg_n, Sect. 3.4.2

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