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Bodies of Constant Width

An Introduction to Convex Geometry with Applications

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Preface

This book might well have been titled *Some Geometry and Convexity* instead of *Bodies of Constant Width*.

Convexity is a small branch of classical mathematics located at the confluence of geometry, analysis, and combinatorics. Its origins can be traced back to Archimedes. In his works *On the Sphere and the Cylinder* and *On Plane Equilibriums* he defined the concepts, which we call today curves and convex surfaces. In these writings he postulated that every convex body contains its center of mass, a claim which was proved much later by H. Minkowski as a generalization of Cauchy's well-known formulas for calculating perimeters and areas of curves and convex surfaces using the lengths and areas of their projections.

The first mathematician who studied sets, curves, and surfaces characterized solely by their convexity properties was H. Brunn, in two articles published in 1887 and 1889: *On ovals and convex surfaces* and *On curves without inflection points*. These articles contained many results, some of them intuitively obvious, presented without proofs and with less rigor than would be demanded today, but in many cases quite deep. It was H. Minkowski who, appreciating the originality and profound nature of Brunn's results, fleshed them out and shaped them into what is known today as the Brunn–Minkowski Theory. Along with results by A. D. Aleksandrov in the 1930s, this work ushered in the modern theory of convexity. Convexity has broadened considerably since then, new areas have opened up, and others "once forgotten" have been revitalized; examples are combinatorial aspects of convexity, the theory of convex polyhedra, and the local theory of Banach spaces. In addition, convexity has had a great influence on applications by way of its relationship to optimization and linear programming.

The majority of books and expository articles on convexity mention bodies of constant width in one chapter or section. The authors found themselves falling under the spell of the mysterious beauty of bodies of constant width, and planned this book thinking in the transversality of the topic. Yet, in the writing, the subject, having started as bodies of constant width, became convexity. This book here had an antecedent, namely, a textbook published in Spanish "25 years ago" by one of the authors. But this textbook covered only some aspects of bodies of constant width. The present book deals with the most classical and representative results and techniques of standard convexity, including mixed volumes, spherical integration, Cauchy formulas, and others. Hence it might be considered as a textbook, but it also covers, mainly due to the notes at the end of each chapter, the existing material about bodies of constant width. Furthermore, the set of exercises at the end of every chapter reinforces both vocations because some of the exercises are simple and others can demand some research effort. We subdivided them into three classes regarding the degree of difficulty; they are correspondingly marked by stars (the most difficult exercises are marked with two stars).

Curves of constant width and their properties have been known for centuries. Leonhard Euler, in fact, studied them under the name "orbiforms" from the Latin word for circle-shaped curves. Euler was interested in figures of constant width, whose boundaries could be represented as the evolute of a hypocycloid. Nearly, a hundred years later, in 1875, Franz Reuleaux published a book on kinematics in which he mentioned curves of constant width and gave some examples. He later gave the construction of what might be considered as the simplest constant width curve which is not a circle, and today bears his name. Interest in bodies of constant width grew significantly near the beginning of the twentieth century. H. Minkowski, A. Hurwitz, and "shortly thereafter" E. Meissner were among the first who contributed to the area. In 1911, the Schilling Verlag [1030] published a collection of mathematical models, which included some constant width curves and models of constant width bodies, molded in plaster and inspired by Meissner's examples. The list of further renowned mathematicians who have helped to extend the theory of constant width shapes contains the names of W. Blaschke, H. Lebesgue, K. Reidemeister, and, more recently, V. G. Boltyanski, A. S. Besicovitch, G. D. Chakerian, H. Groemer, and R. Schneider. It is evident from the number of recent research articles on bodies of constant width and closely related notions that the field is increasing.

There exists a broad, diverse body of knowledge on bodies of constant width supported by an extensive, sophisticated theoretical framework. Many famous mathematicians have worked in the area, and the success of their popularization is due to the fascinating geometric nature of the topic. It is surprising, therefore, that this may be the first book ever dedicated exclusively to bodies of constant width. It is the hope of the authors that this book fulfills its goal as a textbook on geometry and convexity, but furthermore, that it is successful in conveying the attraction that constant width bodies have for those who fall under their spell.

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