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Differential Equations on Measures and Functional Spaces

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Preface

Objectives, scope and methodology

This is an advanced text on ordinary differential equations (ODEs) in Banach and more general locally convex spaces, most notably ODEs on measures and various function spaces. The methodology is carefully chosen in order to provide a very concise introduction of the fundamentals and then move on quickly, but rigorously and systematically to the up-fronts of modern research in linear and nonlinear PDEs, ΨDEs, general kinetic equations and fractional evolutions. More than half of the book content has not previously been included in any textbook. Other parts have been streamlined and given unified arguments.

The level of generality was chosen such that the book content is suitable for the study of the most important nonlinear equations of mathematical physics, such as Boltzmann, Smoluchovskii, Vlasov, Landau–Fokker–Planck, Cahn–Hilliard, Hamilton–Jacobi–Bellman, nonlinear Schrödinger or McKean–Vlasov diffusions and their nonlocal extensions, of mass-action-law kinetics from chemistry, as well as of nonlinear evolutions arising in evolutionary biology, mean-field games, optimization theory, epidemics and system biology, in general models of interacting particles or agents that describe splitting and merging, collisions and breakage, mutations or the preferential-attachment growth on networks. With this objective in mind, the abstract vector spaces are introduced and studied mostly not for their own sake, but as a convenient tool for storing and summarizing the basic properties of the concrete infinite-dimensional spaces of smooth or integrable functions, measures and distributions (generalized functions), which are crucial for the above-mentioned equations. In other words, the general theory is developed as a tool for effectively solving concrete problems, and it aims at simplifying and not complicating the matter. In accordance with this approach, we are not dealing much with 'pathologies' that may arise in abstract spaces, but rather focus on the regularity properties of the most important classes of equations.

A large number of remarks and comments are scattered throughout the text which stress the interconnections between various parts of the book and aim at revealing where and how a particular result is used in other chapters, or may be used in other contexts.

In order to make the text appealing and accessible to readers with different backgrounds, much attention is paid to the clarification of the links between the languages of pseudo-differential operators (ΨDOs), generalized functions, operator theory, abstract linear spaces, fractional calculus and path integrals. With the same objective in mind, lots of attention is paid to proper definitions of all the objects that are used. Some definitions are even repeated in different chapters. A detailed subject index refers to the pages where the corresponding notions are defined. Also, the book contains many exercises that deal with examples and further developments. Note that these exercises never substitute the proofs of the main results. Solutions are provided for most exercises of the four initial chapters. Exercises in later chapters are more research-oriented.

General context and specific features

The basic classes of partial differential, integral and pseudo-differential equations usually lead to ODEs in infinite-dimensional spaces with unbounded, not Lipschitz-continuous and/or singular coefficients. Roughly speaking, our major tools in their study derive from the methods of semigroups and propagators, on the one hand, and from the exploitation of some kind of positivity preservation, on the other.

The overall emphasis is on the well-posedness of the problems (existence, uniqueness and continuous dependence of solutions on the initial data), sensitivity (smooth dependence of solutions on initial data and/or parameters), regularity of the solutions in various classes of smooth functions equipped with pointwise or integral norms, with precise growth estimates, and either the integral representations of solutions (whenever possible) or the natural approximating schemes that allow for various kinds of numerical algorithms to be employed for finding solutions. Apart from being crucial for numeric computations, explicit estimates are important for studying equations with random coefficients, which requires precise control over all bounds. Together with the regular solutions, various concepts of generalized solutions are introduced, whereby two basic classes of such solutions are stressed: generalized solutions by approximations (that can be approximations of regular solutions or approximations by discrete times) and generalized solutions by duality (that can be a Banach-space duality or, more generally, duality of locally convex spaces, the most notable example of the latter being the method of generalized functions).

A unique feature of the exposition in this book is that it is strongly influenced by the links with probability theory and Markov processes (Feller semigroups, Lévy–Khintchin generators, path integrals are standard players in stochastic analysis, but not in standard texts on ODEs), though always remaining independent of these links. The links are crucial for modern developments in the field, since probability theory keeps steadily penetrating all areas of natural and even social sciences. They are made explicit in several side notes that are aimed at readers with some knowledge of stochastic analysis. The links are revealed in detail in [147], [148]. Other accessible books on the links between PDE and stochastics are [12], [68] and [118].

The exposition is also strongly influenced by the fractional calculus, which is rapidly developing as an appropriate tool to deal with various complex problems in natural and social sciences, see, e.g., [253, 255, 263]. Although results on fractional differential equations are analysed in special sections (that readers may choose to omit), it turns out that the type of singularities occurring in fractional equations are in fact quite common in other settings, like nonlinear diffusions or general perturbation theory estimates. Therefore, the growth estimates of the solutions and their sensitivity to parameters are naturally expressed in terms of the Mittag-Leffler functions in various, seemingly unrelated contexts, these functions being the main players in fractional calculus. A unified abstract framework for these contexts makes it possible to treat them in a very effective and concise way.

The source of many developments in the theory of differential equations (especially nonlinear differential equations) can be traced back to the analysis of systems of interacting particles (or agents in the social context) in the limit of large particle numbers (dynamic law of large numbers). Although this link has not been formally developed in the book, it was crucial for the selection of material and methodology.

Another source of new methods and ideas in nonlinear differential equations is the theory of optimization and competitive control systems. This link is being developed here, including the analysis of various classes of the Hamilton–Jacobi–Bellman equation, forward-backward systems (occurring in mean-field games), the Riccati equation and the replicator dynamics of evolutionary game theory and controlled systems of interacting agents.

Since differential equations are a key tool in almost all developments and applications of mathematics, many introductory textbooks on differential equations are available. Traditionally, the topic is included in the undergraduate curriculum in two separate parts: ordinary differential equations (ODEs) and partial differential equations (PDEs). Examples for classical texts on ODEs are [15, 104, 220, 224]. Meanwhile, the standard theory of PDEs is much more diversified. Starting from the classical second-order equations of mathematical physics (Laplace, heat and wave equations), it utilizes various methods for various types of equations. Therefore, the boundaries of even the core of the subject are difficult to overview, and the same applies for a comprehensive list of textbooks. However, a large portion of these methods can be unified by looking at partial differential operators as representatives of linear operators in certain abstract spaces, and then applying the tools of functional analysis. In this framework, PDEs – and more general integro-differential and pseudo-differential equations – are considered ODEs in abstract linear spaces, and the boundary between these two parts of the theory is melting away. The well-known book [219] was one of the first systematic developments of this framework. Following this general idea, the present book provides a unique concise and application-oriented exposition of ODEs on measures and functional spaces, starting from scratch and moving up to the level of modern research in many directions, including non-equilibrium statistical mechanics, nonlinear quantum mechanics, fractional evolutions, evolutionary biology, models of interacting agents, and others.

Readers and prerequisites

This textbook is designed to serve as a multi-purpose learning resource on differential equations. It is mostly aimed at postgraduate or final year undergraduate courses for mathematics students. However, the final Chapters can also be of interest to researchers in linear and nonlinear differential equations. On the other hand, the book can also be used for basic undergraduate courses and self-studies. For instance, Chapter 2 can be considered an intensive undergraduate introductory course on ODEs, if the term ‘Banach space’ is substituted by ‘Euclidean space $\mathbf{R}^d$’ and if the integration methods of the simplest one-dimensional equations are added as exercises. A study in this framework would help the students to grasp the more abstract approaches from the beginning of their curriculum, which makes their further transition to graduate courses more easy. Similarly, Chapter 3 on the equations in $\mathbf{R}_+^d$ and l_1^+ does not require any prerequisites apart from introductory calculus and linear algebra.

The overall level of presentation is meant to be appropriate for readers who are familiar with basic calculus and linear algebra, with the principles of convergence in general compact, metric and Banach spaces, with the basic notions of linear spaces, linear operators, dual spaces and operators, with the theory of measure, integration and L_p-spaces and occasionally with some functions of complex variables. No prior knowledge of ODEs is assumed.

Each of the first four chapters can serve as the basis for a crash course on their respective topic. Put together, they cover a range of topics that is appropriate for a full one semester module. Elements of the other chapters can be used to enhance the course in various particular directions, or as a basis for more advanced courses. For each chapter, a specific abstract and summary are provided.

The material of the last chapters has been adapted from research articles and specialized monographs. Their topical selection was of course influenced by the research interests of the author, but a strong attempt was made to choose only topics within the mainstream of research, including general methods which can be used in a variety of developments and which at the same time have grown mature enough to be presented in a more or less final form.

Though aimed at mathematicians and filled with abstract theory, the book is meant to be truly application-oriented: Not in the sense that it is to be used for producing certain concrete industrial products, but in the sense that the abstract theory is developed in order to effectively solve basic concrete problems that arise in natural sciences or modelling of social processes – that is, as a tool to streamline and simplify the analysis of these problems, and not for the sake of generality in its own right.

Bibliographic comments

Due to the immense amount of literature on the topics that are touched upon in the book, it was unfortunately impossible to provide an exhaustive guide to all relevant contributions. Instead, the given bibliography essentially includes the sources that have been used by the author for the preparation of the manuscript, as well as some classical textbooks and key references for related further developments.

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Standard notations

$\mathbf{N}$, $\mathbf{Z}$, $\mathbf{R}$, $\mathbf{C}$ Sets of natural, integer, real and complex numbers

$\mathbf{Z}_+$ $\mathbf{N} \cup \{0\}$

$\mathbf{R}_+$ $\{x \in \mathbf{R} : x \geq 0\}$

$\mathbf{N}^\infty$, $\mathbf{Z}^\infty$, $\mathbf{R}^\infty$, $\mathbf{C}^\infty$ Sets of sequences from $\mathbf{N}$, $\mathbf{Z}$, $\mathbf{R}$, $\mathbf{C}$

$\mathbf{Z}_+^\infty, \mathbf{R}_+^\infty$ Subsets of $\mathbf{Z}^\infty, \mathbf{R}^\infty$ with non-negative elements

$\mathbf{C}^d$, $\mathbf{R}^d$ Complex and real d-dimensional spaces

(x, y) or xy Scalar product of the vectors $x, y \in \mathbf{R}^d$; also $x^2 = |x|^2 = (x, x)$

$|x|$ or $\|x\|$ Standard Euclidean norm $\sqrt{(x,x)}$ of $x \in \mathbf{R}^d$ or $x \in \mathbf{C}^d$

$\operatorname{Re} a, \operatorname{Im} a$ Real and imaginary part of a complex number a

$[x]$ Integer part of a real number x (maximal integer not exceeding x)

$\operatorname{sgn} x = \operatorname{sgn}(x)$ The sign of x (equals $1, 0, -1$ if $x < 0, x = 0, x > 0$ respectively)

S^d Unit sphere in $\mathbf{R}^{d+1}$

$C(X)$ For a metric or topological space X, the Banach space of bounded continuous functions on X equipped with the sup-norm $\|f\| = \|f\|_{C(X)} = \sup_{x \in X} |f(x)|$

$\mathcal{M}(X)$ Banach space of finite signed Borel measures on X

$\mathcal{M}^+(X)$ and $\mathcal{P}(X)$ The subsets of $\mathcal{M}(X)$ of positive and positive normalized (probability) measures

$C_\infty(X)$ For a locally compact X, the subspace of $C(X)$ consisting of functions that tend to zero at infinity

$C_f(X)$ For a positive function f on X, the Banach space of continuous functions on X with a finite norm $\|g\|_{C_f(X)} = \|g/f\|_{C(X)}$

$\mathcal{M}_f(X)$ For a positive continuous function f on X, the space of Borel measures on X with a finite norm $\|\mu\|_{\mathcal{M}_f(X)} = \sup\{(g, \mu) : \|g\|_{C_f(X)} \leq 1\}$

$C_{f,\infty}(X)$ For a locally compact X and a positive function f, the subspace of $C_f(X)$ consisting of functions g such that $\frac{g(x)}{f(x)} \to 0$, as $x \to \infty$

$C^k(\mathbf{R}^d)$ or short C^k Banach space of k times continuously differentiable functions with bounded derivatives on $\mathbf{R}^d$, with the norm being the sum of the sup-norms of the function itself and all its partial derivatives up to and including the order k

$C^k_\infty(\mathbf{R}^d) \subset C^k(\mathbf{R}^d)$ Functions whose derivatives up to and including order k are all in $C_\infty(\mathbf{R}^d)$

$\nabla f = (\nabla_1 f, \ldots, \nabla_d f) = \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_d}\right)$ The gradient of the function f

$\Delta = \nabla^2 = \sum_j \nabla_j^2$ The Laplacian operator

$\nabla^{\otimes 2} f = \frac{\partial^2 f}{\partial x^2}$ The matrix of the second-order derivatives of f, sometimes referred to as the Hessian

$L_p(X,\mu)$ or $L^p(X,\mu)$, $p \geq 1$ The Banach spaces of (the equivalence classes of) integrable functions on the metric or topological space X with respect to the Borel measure μ, equipped with the p-norm $\|f\|_p = \left[\int |f(x)|^p \, \mu(dx)\right]^{1/p}$

$L_p(\mathbf{R}^d)$ The space $L_p(\mathbf{R}^d, \mu)$ with Lebesgue measure μ

$S(\mathbf{R}^d)$ Schwartz space of fast-decreasing functions: $\{f \in C^\infty(\mathbf{R}^d) : \forall k, l \in \mathbf{N},\ |x|^k \nabla^l f \in C_\infty(\mathbf{R}^d)\}$

$|\nu|$ The (positive) total variation measure for a signed measure ν

$(f,g) = \int f(x)g(x)\,dx$ Scalar product for functions f, g on $\mathbf{R}^d$. For $f \in C(X)$, $\mu \in \mathcal{M}(X)$, the following notation is used: $(f,\mu) = (\mu, f) = \int_X f(x)\mu(dx)$

A^T or A' Transpose of a matrix A

$A^\star$ or A' Dual or adjoint operator of A

$\ker A$, $\operatorname{tr} A$ Kernel and trace of the matrix A

$\mathbf{1}_M$ Indicator function of a set M (equals one or zero according to whether its argument is in M or not)

$\mathbf{1}$ Constant function equal to one, and also the identity operator

$f = O(g)$ For functions f and g, this means that $|f| \leq Cg$ for some constant C

$f = o(g)_{x \to a}$ For functions f and g, this means that $\lim_{x \to a} \frac{f(x)}{g(x)} = 0$

Standard abbreviations

ODE Ordinary differential equation

PDE Partial differential equation

ΨDE Pseudo-differential equation

ΨDO Pseudo-differential operator

r.h.s., l.h.s. Right-hand side and left-hand side, respectively