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Miroslav Josipović

Geometric Multiplication of Vectors

An Introduction to Geometric Algebra in Physics

Birkhäuser

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Additional material to this book can be downloaded from http://extras.springer.com.

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So God made the people speak many different languages. . .

Virus

The ***World Health Organization*** has announced a worldwide epidemic of the ***coordinate virus*** in mathematics and physics courses at all grade levels. Students infected with the virus exhibit compulsive ***vector avoidance*** behavior, unable to conceive of a vector except as a list of numbers, and seizing every opportunity to replace vectors by coordinates. At least two-thirds of physics graduate students are severely infected by the virus, and half of those may be so permanently damaged that they will never recover. The most promising treatment is a strong dose of ***geometric algebra***. (*Hestenes*)

Cat

*When the spiritual teacher and his disciples began their evening meditation, the cat who lived in the monastery made so much noise that it distracted them. Therefore, the teacher ordered tying up the cat during the evening practice. Years later, when the teacher died, tying the cat continued during meditation sessions. When the cat died, another cat was brought to the monastery and tied up. Centuries later, a learned descendant of the spiritual teacher wrote a scholarly treatise on the religious significance of tying up a cat during meditation. (*Zen story*)*

Empty Your Cup

A university professor went to visit a famous Zen master. While the master quietly served tea, the professor talked *about Zen. The master filled the visitor's cup to the brim, and then kept pouring. The professor watched the overflowing cup until he could no longer restrain himself.* It is overfull! No more will go in! *the professor expostulated.* You are like this cup*, the master replied,* how can I show you Zen unless you first empty your cup*? (*Zen story*)*

To my sisters

Preface

The purpose of this text is to introduce the interested reader to the world of geometric algebra. Why?

All right, imagine Neelix and Tuvok (Tuvok is a Vulcan from the starship Voyager) are engaged in conversation. Neelix hopes to sell a new product to Tuvok. His intention is to intrigue Tuvok quickly, giving him as little information as possible, and the ultimate goal is that Tuvok, after using it, will be surprised by the quality of the product and recommend it to others. Do not forget that Tuvok can barely tolerate Neelix's nonstop blathering. Let us begin.

Neelix Mr. Vulcan, would you like to rotate objects without matrices, in any dimension?

Tuvok Mr. Neelix, do you offer me quaternions?

Neelix No, they work in 3D only. I have something much better. In addition, you will be able to calculate with spinors as well.

Tuvok Spinors? Come on, Mr. Neelix, you're not trying to say that I will be able to work with complex numbers, too?

Neelix Yes, Mr. Vulcan, the whole of complex analysis, generalized to higher dimensions. And you will get rid of tensors.

Tuvok Excuse me, what? I'm a physicist… it is impossible…

Neelix It is indeed possible. You do not need coordinates. Moreover, you will be able to do the special theory of relativity and quantum mechanics using the same tool. And all integral theorems that you recognize, including those in the complex field, become a single theorem.

Tuvok Come on… nice idea… I work a lot with Lie algebras and groups…

Neelix In the package…

Tuvok Are you kidding me, Mr. Neelix? Ok, let's suppose that I believe you. How much would that product cost me?

Neelix Pennyworth, Mr. Vulcan; the only price is to multiply vectors differently.

Tuvok That's all? You are offering me all of this for such a small price? What's the trap?

Neelix There isn't one. Except that you will have to spend some time to learn how to use the new tool.

Tuvok Time? I just do not have… Besides, why would I ever give up coordinates? You know, I am quite adept at juggling indices. I have my career…

Neelix Do the physical processes you are studying depend on the coordinate systems you choose?

Tuvok I hope not.

Neelix There. Does a rotation by matrices provide you a clear geometric meaning when you do it?

Tuvok No. I need to make an effort to discover one.

Neelix Now you will not have to for the most part. It will be available to you at each step.

Tuvok Mr. Neelix, I'm curious. Where did you get this new tool?

Neelix Well, Mr. Vulcan, it is an old tool from the Earth, nineteenth century, I think, invented by humans Grassmann and Clifford.

Tuvok What? How is that I am not familiar with it? Isn't that strange?

Neelix Well, I believe that human Gibbs and his followers had a hand in it. Allegedly, human Hestenes was trying to tell the other humans about it, but they did not pay any attention. You will agree, Mr. Vulcan, that humans are really funny sometimes.

Tuvok Mr. Neelix, this is a once in a blue moon case in which I just have to agree with you.

Tuvok buys the product and lives long and prospers. Then, of course, he recommends the new tool to the captain…

This text is motivational, so that the reader can see "what's up." It introduces basic concepts and provides some insights into possible applications. We endeavored to choose simple examples and solved problems; the reader will also find problems to solve. An active reading of the text is suggested, with paper and pencil in hand. Once you achieve some clear insights, computers will help in calculations and visualization. There is a great deal of literature, as well as computer programs, available on the Internet; the reader should do some research in this direction. Some facts are intentionally repeated throughout the text, from slightly different points of view, just to help novices to become comfortable with concepts and relationships. There are grounds to believe that geometric algebra is the mathematics of the future. Paradoxically, although formulated in the mid-nineteenth century, it was mainly ignored, due to a range of (unfortunate) circumstances. This book is intended primarily for young people; those with established careers probably will not easily accept something new. A background in physics and mathematics at the undergraduate level is welcome for some parts of the text. However, it is possible to follow the text using the Internet to find explanations for less familiar terms. Advanced high-school students should be able to read at least parts of the text. A useful source is the book [1]; it can help those who are beginning with algebra and geometry. The book [2] is rather difficult; we recommend it to those who want to do some serious thinking. We also recommend Hestenes's articles.

The reader should adopt the idea that vector multiplication as presented here is natural and justified; what follows are the consequences of such multiplication. The reader can come up independently with arguments to account for the definition of the geometric product. My intention is for the reader to accept that the geometric product is not just a "neat trick." It arises naturally from the concept of a vector. That fact changes a great deal of mathematics. A simple rule that parallel vectors commute, while orthogonal vectors anticommute, produces an incredible amount of mathematics, uniting various mathematical disciplines into the universal language of geometric algebra. Finally, readers well disposed toward geometry may find a great deal of pleasure in geometric algebra. Objects like bivectors are not just new and interesting; they are magical, and they evoke juvenescence and intuitive conceptions about mathematics. In fact, in this book we are trying to explain a new concept of numbers and a special language in which such numbers represent oriented geometric objects. Such oriented numbers are intuitive and can replace unintuitive objects like matrices and tensors, giving us, along the way, universality and great possibilities for the unification of different branches of mathematics.

In the text, we treat for the most part the three-dimensional (3D) Euclidean vector space, and we do so for three main reasons. First, generalizations are usually straightforward; second, the reader can rely on a powerful geometric intuition; and finally, the 3D Euclidean vector space has a lot to offer in the new mathematical language. Its beauty and richness are hidden behind the veil of traditional mathematics.

The Mathematica users can find a special implementation of *Cl*3 at official web address from the Publisher.

You are welcome to send comments or questions to me at miroslav.josipovic@gmail.com

Zagreb, Croatia Miroslav Josipović
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References

1. J. Vince, *Geometric Algebra for Computer Graphics* (Springer, London, 2008)
2. D. Hestenes, G. Sobczyk, *Clifford Algebra to Geometric Calculus* (Reidel, Dordiecht, 1984)

The original version of this book was revised. An correction to this book can be found at https://doi.org/10.1007/978-3-030-01756-9_7

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Contents

About the Author

Miroslav Josipović is a Croatian physicist and musician with special interest in revising the language of mathematical physics and in creating new ways of teaching physics. Revision of the language of mathematical physics can be achieved by relying on geometric algebra, a powerful language of physics and mathematics that brings geometric clarity and unseen possibilities of unification and generalizations. Experiments with physics teaching lead the author to the belief that students can understand nontrivial concepts of physics at an early age, but teachers have to give up the teaching of formalism and offer living examples that encourage curiosity. The feeling that it is possible to understand motivates the students greatly, and the job of a teacher is to enable young people to attain such a feeling. If they are successful, elementary school pupils will be able to solve difficult conceptual problems related to Newton's first law, for example, or high-school students will be able to deal with problems of the special theory of relativity that are generally considered to be difficult at that age. Young students are smart. Mathematics has to rely on geometric intuition, and the author firmly believes that geometric algebra can be gradually introduced as early as elementary school, first through games with oriented geometric objects, and later by extending the concept of a vector to oriented surfaces (bivectors) and the oriented volumes (3-vectors).

Mathematical Notation

$x, \vec{x}, \mathbf{x}$	a vector
$\lvert\mathbf{x}\rvert$	the magnitude (length) of the vector $\mathbf{x}$
$\lVert A \rVert$	the norm of the element A, defined by the scalar product
$x \propto y$	x is proportional to y
$\equiv$	equal to (by definition)
AB	the geometric product (GP)
$A \cdot B$	the inner product
$A * B$	the scalar product
$A \bullet B$	the dot product
$A \wedge B$	the outer (wedge) product
$A \vee B$	the regressive product
$a \times b$	the cross product (for vectors in 3D)
$A \rfloor B$	the left contraction (LC)
$[A, B]$	the commutator of A and B, $(AB - BA)/2$
$\{A, B\}$	the anticommutator of A and B, $(AB + BA)/2$
$\mathrm{P}_B(A)$	a projector of A onto the subspace characterized by B
Cln	the Clifford (geometric) algebra in n-dimensional Euclidean space
e_i	a vector of a basis (usually orthonormal, in $Cl3$ we sometimes use σ_i)
e^i	a vector of a reciprocal basis, $e^i \cdot e_j = \delta_{ij}$
$\hat{\sigma}_i$	the ith Pauli matrix
$e_i e_j \equiv e_{ij}$	a unit bivector (in an orthonormal basis e_i)
$j = e_{123}$	the unit pseudoscalar in $Cl3$
k-vector	single graded element of the grade k
$M^{\Delta} = M \rfloor I^{-1}$	the dual operation, I is the unit pseudoscalar
M^*	complex conjugation, e.g., $(\alpha + \mathbf{n}j + \beta j)^* = \alpha - \mathbf{n}j - \beta j,\ \alpha, \beta \in \mathbb{R}$
$\langle M \rangle_r$	the grade r
$\langle M \rangle = \langle M \rangle_0$	the grade 0
$M^\dagger$	the reverse involution
$\overleftarrow{M}$	the grade involution ($\hat{M}$ is common; however, we use the hat for unit elements)
$\bar{M}$	the Clifford involution (conjugation, main involution)
$\langle M \rangle_R = (M + M^\dagger)/2$	the real part of a multivector (*paravector*) in $Cl3$
$\langle M \rangle_I = (M - M^\dagger)/2$	the imaginary part of a multivector in $Cl3$
$\langle M \rangle_S = (M + \bar{M})/2$	the scalar part of a multivector (complex scalar, $\alpha + \beta j$) in $Cl3$, $\alpha, \beta \in \mathbb{R}$
$\langle M \rangle_V = (M - \bar{M})/2$	the vector part of a multivector (complex vector, $\mathbf{x} + \mathbf{n}j$) in $Cl3$

$\langle M\rangle_{+}=\left(M+\overleftarrow{M}\right)/2$	the even part of a multivector ($\alpha + \mathbf{n}j$) in $Cl3$, $\alpha \in \mathbb{R}$
$\langle M\rangle_{-}=\left(M-\overleftarrow{M}\right)/2$	the odd part of a multivector ($\mathbf{x} + \beta j$) in $Cl3$, $\beta \in \mathbb{R}$
$\lvert M\rvert=\sqrt{M\bar{M}}$	the multivector amplitude (MA; sometimes we use it as $M\bar{M}$)
δ_{ij}	the Kronecker delta symbol, $\delta_{ij}=1$ for $i=j$ and $\delta_{ij}=0$ for $i\neq j$
$\partial_k \equiv \frac{\partial}{\partial x^k}$	a partial derivative, $\frac{\partial x^i}{\partial x^j}=\delta_{ij}$
$\nabla = e^k\partial_k$	the differential operator *nabla* (summation over $k=1$, 2, 3)
$\partial = \partial_t + \nabla$	a differential operator as a *paravector*
$\mathfrak{R}^{(p,q,r)}$	a vector space with the *signature* (p, q, r)
$\mathbb{R}$	the set of real numbers ($\alpha, \beta, \ldots$)
i	the imaginary unit, $i=\sqrt{-1}$
$\mathbb{C}$	the set of complex numbers in $Cl3$, such as $\alpha + \beta j$; sometimes $\alpha+\beta i,\ \ i=\sqrt{-1},\ \ \alpha,\beta\in\mathbb{R}$
F	a linear transformation (note the text format)
$\bar{\mathrm{F}}$	adjoint (transpose) of the linear transformation F
$\lvert\psi\rangle$	a vector in a complex vector space, *ket* (a column vector, usually represents spinors)
$\langle\psi\rvert$	the Hermitian conjugate (transposed and complex conjugated) of $\lvert\psi\rangle$, *bra* (a row vector)
$\langle\phi\lvert\psi\rangle$	the inner product in a complex vector space
$\lvert\psi\rangle\langle\phi\rvert$	the outer product in a complex vector space
$\langle\phi\lvert O\rvert\psi\rangle$	the *expectation value* of the *operator* O in a complex vector space

Acronyms

APS	algebra of physical space, Pauli algebra, $Cl3$
CGA	the *conformal model* in geometric algebra
ICS	inertial coordinate system
IRF	inertial reference frame
GA	geometric algebra
GP	geometric product
LC	left contraction
MA	multivector amplitude
STA	spacetime algebra
STR	the special theory of relativity

Reference Labels

E*n*	the *n*th task with a solution in Solutions (Sect. 6.1)
Fig.*n*	Figure *n* from the text
L*n*	the link *n* from the References and Links to specific subjects
n	problem *n* in Problems (Sect. 6.2)
[*n*]	reference *n*
R*n*	reference *n* from the References and Links to specific subjects