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Free Probability and Random Matrices

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Dedicated to Dan-Virgil Voiculescu,
who gave us freeness, and Betina and Jill,
who gave us the liberty to work on it.

Contents

Introduction

This book is an invitation to the world of free probability theory.

Free probability is a quite young mathematical theory with many avatars. It owes its existence to the visions of one man, Dan-Virgil Voiculescu, who created it out of nothing at the beginning of the 1980s and pushed it forward ever since. The subject had a relatively slow start in its first decade but took on a lot of momentum later on.

It started in the theory of operator algebras, showed its beautiful combinatorial structure via non-crossing partitions, made contact with the world of random matrices, and reached out to many other subjects like representation theory of large groups, quantum groups, invariant subspace problem, large deviations, quantum information theory, subfactors, or statistical inference. Even in physics and engineering, many people have heard of it and find it useful and exciting.

One of us (RS) has already written, jointly with Alexandru Nica, a monograph [137] on the combinatorial side of free probability theory. Whereas combinatorics will also show up in the present book, our intention here is different: we want to give a flavour of the breadth of the subject; hence this book will cover a variety of different facets, occurrences, and applications of free probability; instead of going in depth in one of them, our goal is to give the basic ideas and results in many different directions and show how all this is related.

This means that we have to cover subjects as different as random matrices and von Neumann algebras. This should, however, not to be considered a peril but as a promise for the good things to come.

We have tried to make the book accessible to both random matrix and operator algebra (and many more) communities without requiring too many prerequisites. Whereas our presentation of random matrices should be mostly self-contained, on the operator algebraic side, we try to avoid the technical parts of the theory as much as possible. We hope that the main ideas about von Neumann algebras are comprehensible even without knowing what a von Neumann algebra is. In particular, in Chapters 1–5, no von Neumann algebras will make their appearance.

The book is a mixture between textbook and research monograph. We actually cover many of the important developments of the subject in recent years, for which no coherent introduction in monograph style has existed up to now.

Chapters 1, 2, 3, 4, and 6 describe in a self-contained way the by now well-established basic body of the theory of free probability. Chapters 1 and 4 deal with the relation of free probability with random matrices; Chapter 1 is more of a motivating nature, whereas Chapter 4 provides the rigorous results. Chapter 6 provides the relation to operator algebras and the free group factor isomorphism problem, which initiated the whole theory. Chapter 2 presents the combinatorial side of the theory; as this is dealt with in much more detail in the monograph [137], we sometimes refer to the latter for details. Chapter 3 gives a quite extensive and self-contained account of the analytic theory of free convolution. We put there quite some emphasis on the subordination formulation, which is the modern state of the art for dealing with such questions and which cannot be found in this form anywhere else.

The other chapters deal with parts of the theory where the final word is not yet spoken, but where important progress has been achieved and which surely will survive in one or the other form in future versions of free probability. In those chapters, we often make references to the original literature for details of the proofs. Nevertheless we try also there to provide intuition and motivation for what and why. We hope that those chapters invite also some of the readers to do original work in the subject.

Chapter 5 is on second order freeness; this theory intends to deal with fluctuations of random matrices in the same way as freeness does this with the average. Whereas the combinatorial aspect of this theory is far evolved, the analytic status awaits a better understanding.

Free entropy has at the moment two incarnations with very different flavour. The microstates approach is treated in Chapter 7, whereas the non-microstates approach is in Chapter 8. Both approaches have many interesting and deep results and applications—however, the final form of free entropy theory (hoping that there is only one) still has to be found.

Operator-valued free probability has evolved in recent years into a very powerful generalization of free probability theory; this is made clear by its applicability to much bigger classes of random matrices and by its use for calculating the distribution of polynomials in free variables. The operator-valued theory is developed and its use demonstrated in Chapters 9 and 10.

In Chapter 11, we present the Brown measure, a generalization of the spectral distribution from the normal to the non-normal case. In particular, we show how free probability (in its operator-valued version) allows one to calculate such Brown measures. Again there is a relation with random matrices; the Brown measure is the canonical candidate for the eigenvalue distribution of non-normal random matrix models (where the eigenvalues are not real, but complex).

After having claimed to cover many of the important directions of free probability, we have now to admit that there are at least as many which unfortunately did not make it into the book. One reason for this is that free probability is still very fast evolving with new connections popping up quite unexpectedly.

So we are, for example, not addressing such exciting topics as free stochastic and Malliavin calculus [39, 108, 114], or the rectangular version of free probability [28], or the strong version of asymptotic freeness [48, 58, 88], or free monotone transport [83], or the relation with representation theory [35, 72] or with quantum groups [16, 17, 44, 73, 110, 118, 148], or the quite recent new developments around bifreeness [52, 81, 100, 196], traffic freeness [50, 122], or connections to Ramanujan graphs via finite free convolution [124]. Instead of trying to add more chapters to a never-ending (and never-published) book, we prefer just to stop where we are and leave the remaining parts for others.

We want to emphasize that some of the results in this book owe their existence to the book writing itself and our endeavour to fill apparent gaps in the existing theory. Examples of this are our proof of the asymptotic freeness of Wigner matrices from deterministic matrices in Section 4.4 (for which there exists now also another proof in the book [7]), the fact that finite free Fisher information implies the existence of a density in Proposition 8.18, or the results about the absence of algebraic relations and zero divisors in the case of finite free Fisher information in Theorems 8.13 and 8.32.

Our presentation benefited a lot from input by others. In particular, we like to mention Serban Belinschi and Hari Bercovici for providing us with a proof of Proposition 8.18 and Uffe Haagerup for allowing us to use his manuscript of his talk at the Fields Institute as the basis for Chapter 11. With the exception of Sections 11.9 and 11.10, we are mainly following his notes in Chapter 11. Chapter 3 relied substantially on input and feedback from the experts on the subject. Many of the results and proofs around subordination were explained to us by Serban Belinschi, and we also got a lot of feedback from JC Wang and John Williams. We are also grateful to N. Raj Rao for help with his RMTool package which was used in our numerical simulations.

The whole idea of writing this book started from a lectures series on free probability and random matrices which we gave at the Fields Institute, Toronto, in the fall of 2007 within the Thematic Program on Operator Algebras. Notes of our lectures were taken by Emily Redelmeier and by Jonathan Novak, and the first draft of the book was based on these notes.

We had the good fortune to have Uffe Haagerup around during this programme, and he agreed to give one of the lectures, on his work on the Brown measure. As mentioned above, the notes of his lecture became the basis of Chapter 11.

What are now Chapters 5, 8, 9, and 10 were not part of the lectures at the Fields Institute, but were added later. Those additional chapters cover in big parts also results which did not yet exist in 2007. So this gives us at least some kind of excuse that the finishing of the book took so long.

Much of Chapter 8 is based on classes on “Random matrices and free entropy” and “Non-commutative distributions” which one of us (RS) taught at Saarland University during the winter terms 2013/2014 and 2014/2015, respectively. The final outcome of this chapter owes a lot to the support of Tobias Mai for those classes.

Chapter 9 is based on work of RS with Wlodek Bryc, Reza Rashidi Far, and Tamer Oraby on block random matrices in a wireless communications (MIMO) context and on various lectures of RS for engineering audiences, where he tried to convince them of the relevance and usefulness of operator-valued methods in wireless problems. Chapter 10 benefited a lot from the work of Carlos Vargas on free deterministic equivalents in his PhD thesis and from the joint work of RS with Serban Belinschi and Tobias Mai around linearization and the analytic theory of operator-valued free probability. The algorithms, numerical simulations, and histograms for eigenvalue distributions in Chapter 10 and Brown measures in Chapter 11 are done with great expertise and dedication by Tobias Mai.

There are exercises scattered throughout the text. The intention is to give readers an opportunity to test their understanding. In some cases, where the result is used in a crucial way or where the calculation illustrates basic ideas, a solution is provided at the end of the book.

In addition to the already mentioned individuals, we owe a lot of thanks to people who read preliminary versions of the book and gave useful feedback, which helped to improve the presentation and correct some mistakes. We want to mention in particular Marwa Banna, Arup Bose, Mario Diaz, Yinzheng Gu, Todd Kemp, Felix Leid, Josué Vázquez, Hao-Wei Wang, and Guangqu Zheng.

Further thanks are due to the Fields Institute for the great environment they offered us during the already mentioned thematic programme on operator algebras and for the opportunity to publish our work in their Monographs series. The writing of this book, as well as many of the reported results, would not have been possible without financial support from various sources; in particular, we want to mention a Killam Fellowship for RS in 2007 and 2008, which allowed him to participate in the thematic programme at the Fields Institute and thus get the whole project started, and the ERC Advanced Grant "Non-commutative distributions in free probability" of RS, which provided time and resources for the finishing of the project. Many of the results we report here were supported by grants from the Canadian and German Science Foundations NSERC and DFG, respectively, by Humboldt Fellowships for Serban Belinschi and John Williams for stays at Saarland University, and by DAAD German-French Procope exchange programmes between Saarland University and the Universities of Besançon and of Toulouse.

As we are covering a wide range of topics, there might come a point where one gets a bit exhausted from our book. There are, however, some alternatives, like the standard references [97, 137, 197, 198] or survey articles [37, 84, 141, 142, 156, 162, 164, 165, 183, 191, 192] on (some aspects of) free probability. Our advice: take a break, enjoy those, and then come back motivated to learn more from our book.