Part II
Numerical Analysis

For readers' convenience, we resume here the basic facts from Part I. By recalling the complexification cited in Remark 3.1, we assume $x(t) \in \mathbb{C}^d$ and the state space $X = C([-\tau, 0], \mathbb{C}^d)$. Then, in Part III, we turn back to $C([-\tau, 0], \mathbb{R}^d)$ for applications.

For $L : X \to \mathbb{C}^d$ a linear and bounded functional, the linear autonomous DDE

$$x'(t) = Lx_t, \ t \in \mathbb{R}, \tag{II.1}$$

can be restated as the abstract linear ODE in the infinite dimensional space X

$$u'(t) = \mathscr{A}u(t), \ t \in \mathbb{R},$$

where $u(t) = x_t$ and $\mathscr{A}$ is the infinitesimal generator of the SO-semigroup $\{T(t)\}_{t \geq 0}$. For $t \geq 0$, the (linear and bounded) solution operator $T(t) : X \to X$ is

$$T(t)\varphi = x_t(\cdot; \varphi), \ \varphi \in X,$$

where $x(\cdot; \varphi)$ is the solution of the Cauchy problem for (II.1) with initial function $x_0 = \varphi$ at time $t = 0$. The (linear and unbounded) infinitesimal generator $\mathscr{A} : \mathscr{D}(\mathscr{A}) \subseteq X \to X$ is

$$\begin{cases} \mathscr{D}(\mathscr{A}) = \{\varphi \in X : \varphi' \in X \text{ and } \varphi'(0) = L\varphi\}, \\ \mathscr{A}\varphi = \varphi'. \end{cases}$$

The eigenvalues of $\mathscr{A}$ are the characteristic roots of (II.1) and we are interested in their numerical computation. We present two approaches: the *IG approach*, based on discretizing the infinitesimal generator, and the *SO approach*, based on discretizing an arbitrary solution operator $T(t)$. In the former, we compute numerical approximations of the eigenvalues of $\mathscr{A}$, giving the characteristic roots directly. In the latter, we compute numerical approximations of the nonzero eigenvalues of $T(t)$: being of type $e^{\lambda t}$ for λ a characteristic root, only $\mathrm{Re}(\lambda)$ can be recovered.

Unlike the IG approach, the SO approach can be used on the more general linear nonautonomous DDEs

$$x'(t) = L(t)x_t,\ t \in \mathbb{I}, \tag{II.2}$$

where, for any $t \in \mathbb{I}$, $L(t) : X \to \mathbb{C}^d$ is a linear and bounded functional. In this more general case, the evolution family $\{T(t,\, s)\}_{t \geq s}$ replaces the SO-semigroup. For $t \geq s$ in $\mathbb{I}$, the (linear and bounded) evolution operator $T(t,\, s) : X \to X$ is

$$T(t,\, s)\varphi = x_t(\cdot;s,\, \varphi),\ \varphi \in X,$$

where $x(\cdot;s,\, \varphi)$ is the solution of the Cauchy problem for (II.2) with initial function $x_s = \varphi$ at time $t = s$. The SO approach is used for the numerical computation of the nonzero eigenvalues of an arbitrary evolution operator $T(t,\, s)$ and, in particular, for the computation of the characteristic multipliers of periodic DDEs, i.e., the eigenvalues of $T(\omega,\, 0)$ where ω is the (minimal) period of the function $t \mapsto L(t)$. Eventually, we remark that a similar discretization can be used for computing the Lyapunov exponents for the general case (II.2) as done in [49] (see also [32, 83]).