Graduate Texts in Mathematics 139

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Graduate Texts in Mathematics

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Glen E. Bredon

Topology and Geometry

With 85 Illustrations

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Glen E. Bredon
Department of Mathematics
Rutgers University
New Brunswick, NJ 08903
USA

Editorial Board

S. Axler
Mathematics Department
San Francisco State University
San Francisco, CA 94132
USA

F.W. Gehring
Mathematics Department
East Hall
University of Michigan
Ann Arbor, MI 48109
USA

K.A. Ribet
Department of Mathematics
University of California at Berkeley
Berkeley, CA 94720-3840
USA

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Preface

The golden age of mathematics—that was not the age of Euclid, it is ours.

C.J. KEYSER

This time of writing is the hundredth anniversary of the publication (1892) of Poincaré's first note on topology, which arguably marks the beginning of the subject of algebraic, or "combinatorial," topology. There was earlier scattered work by Euler, Listing (who coined the word "topology"), Möbius and his band, Riemann, Klein, and Betti. Indeed, even as early as 1679, Leibniz indicated the desirability of creating a geometry of the topological type. The establishment of topology (or "analysis situs" as it was often called at the time) as a coherent theory, however, belongs to Poincaré.

Curiously, the beginning of general topology, also called "point set topology," dates fourteen years later when Fréchet published the first abstract treatment of the subject in 1906.

Since the beginning of time, or at least the era of Archimedes, smooth manifolds (curves, surfaces, mechanical configurations, the universe) have been a central focus in mathematics. They have always been at the core of interest in topology. After the seminal work of Milnor, Smale, and many others, in the last half of this century, the topological aspects of smooth manifolds, as distinct from the differential geometric aspects, became a subject in its own right. While the major portion of this book is devoted to algebraic topology, I attempt to give the reader some glimpses into the beautiful and important realm of smooth manifolds along the way, and to instill the tenet that the algebraic tools are primarily intended for the understanding of the geometric world.

This book is intended as a textbook for a beginning (first-year graduate) course in algebraic topology with a strong flavoring of smooth manifold theory. The choice of topics represents the ideal (to the author) course. In practice, however, most such courses would omit many of the subjects in the book. I would expect that most such courses would assume previous knowledge of general topology and so would skip that chapter, or be limited to a brief run-through of the more important parts of it. The section on homotopy should be covered, however, at some point. I do not go deeply into general topology, but I do believe that I cover the subject as completely as a mathematics student needs unless he or she intends to specialize in that area.

It is hoped that at least the introductory parts of the chapter on differentiable manifolds will be covered. The first section on the Implicit Function Theorem might best be consigned to individual reading. In practice, however, I expect that chapter to be skipped in many cases with that material assumed covered in another course in differential geometry, ideally concurrent. With that possibility in mind, the book was structured so that that material is not essential to the remainder of the book. Those results that use the methods of smooth manifolds and that are crucial to other parts of the book are given separate treatment by other methods. Such duplication is not so large as to be consumptive of time, and, in any case, is desirable from a pedagogic standpoint. Even the material on differential forms and de Rham's Theorem in the chapter on cohomology could be omitted with little impact on the other parts of the book. That would be a great shame, however, since that material is of such interest on its own part as well as serving as a motivation for the introduction of cohomology. The section on the de Rham theory of $\mathbf{CP}^n$ could, however, best be left to assigned reading. Perhaps the main use of the material on differentiable manifolds is its impact on examples and applications of algebraic topology.

As is common practice, the starred sections are those that could be omitted with minimal impact on other nonstarred material, but the starring should not be taken as a recommendation for that aim. In some cases, the starred sections make more demands on mathematical maturity than the others and may contain proofs that are more sketchy than those elsewhere.

This book is not intended as a source book. There is no attempt to present material in the most general form, unless that entails no expense of time or clarity. Exceptions are cases, such as the proof of de Rham's Theorem, where generality actually improves both efficiency and clarity. Treatment of esoteric byways is inappropriate in textbooks and introductory courses. Students are unlikely to retain such material, and less likely to ever need it, if, indeed, they absorb it in the first place.

As mentioned, some important results are given more than one proof, as much for pedagogic reasons as for maintaining accessibility of results essential to algebraic topology for those who choose to skip the geometric treatments of those results. The Fundamental Theorem of Algebra is given no less than four topological proofs (in illustration of various results). In places where choice is necessary between competing approaches to a given topic, preference has been given to the one that leads to the best understanding and intuition.

In the case of homology theory, I first introduce singular homology and derive its simpler properties. Then the axioms of Eilenberg, Steenrod, and Milnor are introduced and used exclusively to derive the computation of the homology groups of cell complexes. I believe that doing this from the axioms, without recourse to singular homology, leads to a better grasp of the functorial nature of the subject. (It also provides a uniqueness proof gratis.) This also leads quickly to the major applications of homology theory. After that point, the difficult and technical parts of showing that singular homology satisfies the axioms are dealt with.

Cohomology is introduced by first treating differential forms on manifolds, introducing the de Rham cohomology and then linking it to singular homology. This leads naturally to singular cohomology. After development of the simple properties of singular cohomology, de Rham cohomology is returned to and de Rham's famous theorem is proved. (This is one place where treatment of a result in generality, for all differentiable manifolds and not just compact ones, actually provides a simpler and cleaner approach.)

Appendix B contains brief background material on "naive" set theory. The other appendices contain ancillary material referred to in the main text, usually in reference to an inessential matter.

There is much more material in this book than can be covered in a one-year course. Indeed, if everything is covered, there is enough for a two-year course. As a suggestion for a one-year course, one could start with Chapter II, assigning Section 1 as individual reading and then covering Sections 2 through 11. Then pick up Section 14 of Chapter I and continue with Chapter III, Sections 1 through 8, and possibly Section 9. Then take Chapter IV except for Section 12 and perhaps omitting some details about CW-complexes. Then cover Chapter V except for the last three sections. Finally, Chapter VI can be covered through Section 10. If there is time, coverage of Hopf's Theorem in Section 11 of Chapter V is recommended. Alternatively to the coverage of Chapter VI, one could cover as much of Chapter VII as is possible, particularly if there is not sufficient time to reach the duality theorems of Chapter VI.

Although I do make occasional historical remarks, I make no attempt at thoroughness in that direction. An excellent history of the subject can be found in Dieudonné [1]. That work is, in fact, much more than a history and deserves to be in every topologist's library.

Most sections of the book end with a group of problems, which are exercises for the reader. Some are harder, or require more "maturity," than others and those are marked with a ◆. Problems marked with a ⟡ are those whose results are used elsewhere in the main text of the book, explicitly or implicitly.

Glen E. Bredon

Acknowledgments

It was perfect, it was rounded, symmetrical,
complete, colossal.

MARK TWAIN

Unlike the object of Mark Twain's enthusiasm, quoted above (and which has no geometric connection despite the four geometric–topological adjectives), this book is far from perfect. It is simply the best I could manage. My deepest thanks go to Peter Landweber for reading the entire manuscript and for making many corrections and suggestions. Antoni Kosinski also provided some valuable assistance. I also thank the students in my course on this material in the spring of 1992, and previous years, Jin-Yen Tai in particular, for bringing a number of errors to my attention and for providing some valuable pedagogic ideas.

Finally, I dedicate this book to the memory of Deane Montgomery in deep appreciation for his long-term support of my work and of that of many other mathematicians.

Glen E. Bredon

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