Stochastic Processes: General Theory

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# Stochastic Processes: General Theory

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**To the memory**

**of my parents**

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# Preface

The following work represents a completely revised and enlarged version of my book *Stochastic Processes and Integration* (1979). The new material is so extensive that it was deemed appropriate to modify the title to the current one, which reflects the content and generality. Although the book follows the original format, the changes from the previous edition are everywhere. I shall briefly explain the differences and the additions here.

The present version contains a more detailed and complete treatment of Kolmogorov's existence theorem in terms of projective limits and various applications. The first five chapters are devoted to the general theory of processes, and the final two are largely new. To accommodate all the work and to keep the book in reasonable bounds, several parts of the original presentation have been shortened and some have been omitted.

A few words on the specific changes should contrast the original work with this revised edition. A major difference is the inclusion of a generalized version of Bochner's boundedness principle which enables a novel unification of all the currently used stochastic integrals. This plays a key role in Chapter VI, where both linear and nonlinear higher order stochastic differential equations are presented as applications of this idea. A special feature of these equations, compared to the first order case, is also stressed; and many new problems awaiting solutions

are pointed out. This is not discussed in other books, as far as I know. Chapter VII continues the general theme, but for processes taking values in smooth manifolds or for multiparameters. The old Chapter IV is split into two in the present version, and much of the lifting theory of the original Chapter III is shortened to make room for the new work. A somewhat similar condensation occurs in Chapters I and II of the original version, but the Kolmogorov existence theory of processes is expanded. I tried, however, to present the new version more tightly and it is better focussed. Some suggestions of the reviewers of the original monograph are also taken into account. Since every chapter begins with an outline of its contents, I shall omit further detailed discussion, except noting that complements and exercises parts (often with hints) supplement the text in a number of ways.

Each chapter has a bibliographical notes section assigning proper credits to various contributors. Hopefully I have been successful in this attempt. There is an expanded bibliography as well as notation, author and subject indexes. The numbering system is standard. Thus an item such as VI.3.5 denotes the fifth in Section 3 of Chapter VI. In a given chapter only the section and item number are used, but in a section both the chapter and section numbers are also omitted, retaining only the item number. Further equations are numbered afresh in each section. However, chapter and section titles appear on simultaneous pages to allow easy location of the items.

The revision has taken much more time than originally envisaged. I would like to thank Prof. V.V. Sazonov for some helpful comments and especially for his collaboration on the projective limit theory. A large part of the preparation of the manuscript, using TEX, was accomplished with the great help of Ms. Jan Patterson. I am also grateful to Dr. Y. Kakihara for showing me by example that this TEX craft can be learned by someone like me who is not well-versed in typing. Finally I shall be happy if the subject covered and the problems raised here stimulate enough interest in researchers in furthering the subject.

Riverside, CA. M.M. Rao
May, 1995