Undergraduate Texts in Mathematics

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(continued after Index)

Gerald A. Edgar

Measure, Topology, and Fractal Geometry

With 96 Illustrations and 16 Color Plates

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Preface

What is a fractal? Benoit Mandelbrot coined the term in 1975. There is (or should be) a mathematical definition, specifying a basic idea in geometry. There is also a figurative use of the term to describe phenomena that approximate this mathematical ideal. Roughly speaking, a fractal set is a set that is more "irregular" than the sets considered in classical geometry. No matter how much the set is magnified, smaller and smaller irregularities become visible. Mandelbrot argues that such geometric abstractions often fit the physical world better than regular arrangements or smooth curves and surfaces. On page 1 of his book, *The Fractal Geometry of Nature*, he writes, "Clouds are not spheres, mountains are not cones, coastlines are not circles, and bark is not smooth, nor does lightning travel in a straight line."[**36**, p. 1]*

To define **fractal**, Mandelbrot writes: "A fractal is by definition a set for which the Hausdorff-Besicovitch dimension strictly exceeds the topological dimension."[**36**, p. 15] It might be said that this book is a meditation on that verse. Study of the Hausdorff dimension requires measure theory (Chapter 5); study of topological dimension requires metric topology (Chapter 2). Note, however, that Mandelbrot later expressed some reservations about this definition: "In science its [the definition's] generality was to prove excessive; not only awkward, but genuinely inappropriate ... This definition left out many 'borderline fractals', yet it took care, more or less, of the frontier 'against' Euclid. But the frontier 'against' true geometric chaos was left wide open."[**46**, p. 159] We will discuss in Chapter 6 a way (proposed by James Taylor) to repair the definition. Mandelbrot himself, in the second printing of [**36**, p. 459], proposes "... to use 'fractal' without a pedantic definition, to use 'fractal dimension' as a generic term applicable to *all* the variants in Chapter 39, and to use in each specific case whichever definition is the most appropriate." I have not adopted the first of these suggestions in this book, since a term without a "pedantic definition" cannot be discussed mathematically. I have, however, used the term "fractal dimension" as suggested.

*Bracketed numbers like this refer to the references collected on page 223.

This is a *mathematics* book. It is not about how fractals come up in nature; that is the topic of Mandelbrot's book **[36]**. It is not about how to draw fractals on your computer. (I did have a lot of fun using a Macintosh to draw the pictures for the book, however. There will be occasional use of the Logo programming language for illustrative purposes.) Complete proofs of the main results will be presented, whenever that can reasonably be done. For some of the more difficult results, only the easiest non-trivial case of the proof (such as the case of two dimensions) is included here, with a reference to the complete proof in a more advanced text.

The main examples that will be considered are subsets of Euclidean space (in fact, usually two-dimensional Euclidean space); but as we will see, it is helpful to deal with the more abstract setting of "metric spaces".

This book deals with *fractal geometry.* It does not cover, for example, chaotic dynamical systems. That is a separate topic, although it is related. Another book of the same size could be written* on it. This book does not deal with the Mandelbrot set. Some writing on the subject has left the impression that "fractal" is synonymous with "Mandelbrot set"; that is far from the truth. This book does not deal with stochastic (or random) fractals; their rigorous study would require more background in probability theory than I have assumed for this book.

Prerequisites. (1) Experience in reading (and, preferably, writing) mathematical proofs is essential, since proofs are included here. I will say "necessary and sufficient" or "contrapositive" or "proof by induction" without explanation. Readers without such experience will only be able to read the book at a more superficial level by skipping many of the proofs. (Of course, no mathematics student will want to do that!)

(2) Basic abstract set theory will be needed. For example, the abstract notion of a function; finite vs. infinite sets; countable vs. uncountable sets.

(3) The main prerequisite is calculus. For example: What is a continuous function, and why do we care? The sum of an infinite series. The limit of a sequence. The least upper bound axiom (or property) for the real number system. Proofs of the important facts are included in many of the modern calculus texts, but unfortunately they are often omitted by the instructor (because of student resistance or simple lack of time).

Advice for the reader. Here is some advice for those trying to read the book without guidance from an experienced instructor. The most difficult (and tedious) parts of the book are probably Chapters 2 and 5. But these chapters lead directly to the most important parts of the book, Chapters 3 and 6. Most of Chapter 2 is independent of Chapter 1, so to ease the reading of Chapter 2, it might be reasonable to intersperse parts of Chapter 2 with parts of Chapter 1.

*For example, **[14]**.

In a similar way, Chapters 3, 4, and 5 are mostly independent of each other, so parts of these three chapters could be interspersed with each other.

There are many exercises scattered throughout the text. Some of them deal with examples and supplementary material, but many of them deal with the main subject matter at hand. Even though the reader already knows it, I must repeat: Understanding will be greatly enhanced by work on the exercises (even when a solution is not found). Answers to some of the exercises are given elsewhere in the book, but in order to encourage the reader to devote more work to the exercises, I have not attempted to make them easy to find. When an exercise is simply a declarative statement, it is to be understood that it is to be proved. (Professor Ross points out that if it turns out to be false, then you should try to salvage it.) Some exercises are easy and some are hard. I have even included some that I do not know how to solve. (No, I won't tell you which ones they are.)

Take a look at the Appendix. It is intended to help the reader of the book. There is an index of the main terms defined in the book; an index of notation; and a list of the fractal examples discussed in the text.

Some illustrations are not referred to in the main text. An instructor who knows what they are may use them for class assignments at the appropriate times.

Some of the sections that are more difficult, or deal with less central ideas, are marked with an asterisk (*). They should be considered optional. A section of "Remarks" is at the end of each chapter. It contains many miscellaneous items, such as: references for the material in the chapter; more sophisticated proofs that were omitted from the main text; suggestions for course instructors.

Notation. Most notation used here is either explained in the text, or else taken from calculus and elementary set theory. A few reminders and additional explanations are collected here.

Integers: $\mathbf{Z} = \{\cdots, -2, -1, 0, 1, 2, \cdots\}$.

Natural numbers or positive integers: $\mathbf{N} = \{1, 2, 3, \cdots\}$.

Real numbers: $\mathbf{R} = (-\infty, \infty)$.

Intervals of real numbers:

$$(a, b) = \{\, x : a < x < b \,\};$$
$$(a, b] = \{\, x : a < x \leq b \,\}; \quad \text{etc.}$$

The notation (a, b) also represents an ordered pair, so the context must be used to distinguish.

Set difference or relative complement: $X \setminus A = \{\, x \in X : x \notin A \,\}$.

If $f\colon X \to Y$ is a function, and $x \in X$, I will use parentheses $f(x)$ for the value of the function at the point x; if $C \subseteq X$ is a set, I will use square brackets for the image set $f[C] = \{\, f(x) : x \in C \,\}$.

The union of a family $(A_i)_{i \in I}$ of sets, written

$$\bigcup_{i \in I} A_i,$$

consists of all points that belong to at least one of the sets A_i. The intersection

$$\bigcap_{i \in I} A_i$$

consists of the points that belong to all of the sets A_i. The family $(A_i)_{i \in I}$ is said to be **disjoint** iff $A_i \cap A_j = \varnothing$ for any $i \neq j$ in the index set I.

The **supremum** (or least upper bound) of a set $A \subseteq \mathbf{R}$ is written $\sup A$. By definition $u = \sup A$ satisfies (1) $u \geq a$ for all $a \in A$, and (2) if $y \geq a$ for all $a \in A$, then $y \geq u$. Thus, if A is not bounded above, we write $\sup A = \infty$, and if $A = \varnothing$, we write $\sup A = -\infty$. The **infimum** (or greatest lower bound) is $\inf A$. The **upper limit** of a sequence $(x_n)_{n=1}^{\infty}$ is

$$\limsup_{n \to \infty} x_n = \lim_{n \to \infty} \sup_{k \geq n} x_k.$$

And, if $\alpha(r)$ is defined for real $r > 0$,

$$\limsup_{r \to 0} \alpha(r) = \lim_{s \to 0} \sup_{0<r<s} \alpha(r).$$

Similar notation is used for the **lower limit** or lim inf.

The sign ☺ signals the end of a proof.

The origin of the book. I offered a course in 1987 at The Ohio State University on fractal geometry. It was intended for graduate students in mathematics, and it was based on the books of Hurewicz and Wallman [**30**] and Falconer [**19**]. I tried to keep the prerequisites at a low enough level that, for example, a graduate student in physics could take the course. The prerequisites listed were: metric topology and Lebesgue measure.* When the course was announced, I began getting inquiries from many other types of students, who were interested in studying fractal geometry more rigorously, but did not have even this minimal background. For example, a student in computer science with a strong background in calculus would still have required two more years of mathematics study ("Advanced Calculus" and "Introductory Real Analysis") before being prepared for the course. This book is intended to fit this sort of student. Only a small part of those two courses is actually required for the study of fractal geometry, at least at the most elementary level. The required topics from metric

*Then I found, to my surprise, that Lebesgue integration is not considered necessary for physics students. I suppose the fact that I find this incredible is an illustration of my ignorance of how mathematics is applied in practice.

topology and measure theory are covered in Chapters 2 and 5. (Mathematics students may be able to skip much of these two chapters.)

This book is directly derived from notes prepared for use in a course offered in 1988 in connection with the program for talented high school students that is run here at The Ohio State University by Professor Arnold Ross for eight weeks every summer. The influence of these young students can be seen in many small ways in the book. (In particular, 1.5.7 and 1.6.1.) Past practice in the Ross program suggested the fruitful use of ultrametric spaces.

Parts of the manuscript were read by Manav Das, Don Leggett, William McWorter, Lorraine Rellick, and Karl Schmidt. Their comments led to many improvements in the manuscript. I would like to thank the many people at Springer-Verlag New York, especially mathematics editor Rüdiger Gebauer, mathematics assistant editor Susan Gordon, mathematics editor Ulrike Schmickler-Hirzebruch (editor of the "Undergraduate Texts in Mathematics" series), and production editor Susan Giniger.

The book was prepared by the author on an Apple Macintosh Plus provided by the Department of Mathematics at The Ohio State University. The text was processed using the TEX system (written by Donald Knuth), and the AMS-TEX macro package. The AMS fonts were very useful, in addition to the Computer Modern fonts. TEX was in the form of the public domain programs `ctex` by Tomas Rokicki and `dvi2ps` by Mark Senn and others, both adapted to run under the MPW Shell. The pictures were prepared on the Macintosh using Coral Object Logo, LCSI Logo, MacDraw II, and the Scrapbook. Camera-ready copy was produced on an Apple LaserWriter; the color plates are from color separations produced by MacDraw II.

G. A. Edgar

Columbus, Ohio
March 6, 1990

His days and times are past,
And my reliances on his fracted dates
Have smit my credit: I love and honor him
—W. Shakespeare, *Timon of Athens*

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