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*(continued after index)*

Thomas E. Cecil

# Lie Sphere Geometry

## With Applications to Submanifolds

With 14 Illustrations

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To my sons,

Tom, Mark and Michael

# Preface

The purpose of this monograph is to provide an introduction to Lie's geometry of oriented spheres and its recent applications to the study of submanifolds of Euclidean space. Lie [1] introduced his sphere geometry in his dissertation, published as a paper in 1872, and used it in his study of contact transformations. The subject was actively pursued through the early part of the twentieth century, culminating with the publication in 1929 of the third volume of Blaschke's [1] *Vorlesungen über Differentialgeometrie*, which is devoted entirely to Lie sphere geometry and its subgeometries. After this, the subject fell out of favor until 1981, when Pinkall [1] used it as the principal tool in his classification of Dupin hypersurfaces in $\mathbb{R}^4$. Since that time, it has been employed by several geometers in the study of Dupin, isoparametric and taut submanifolds.

This book is not intended to replace Blaschke's work, which contains a wealth of material, particularly in dimensions two and three. Rather, it is meant to be a relatively brief introduction to the subject, which leads the reader to the frontiers of current research in this part of submanifold theory. Chapters 1 and 2 are accessible to a beginning graduate student who has taken courses in linear and abstract algebra and projective geometry. Chapters 3 and 4 contain the applications to submanifold theory. These chapters require a first graduate course in differential geometry as a necessary background. A detailed description of the contents of the individual chapters is given in the Introduction, which also serves as a survey of the field to this point in time.

I wish to acknowledge certain works which have been especially useful to me in writing this book. Much of Chapters 1 and 2 is based on Blaschke's book. The proof of the Cartan–Dieudonné theorem in Section 2.2 is taken

from E. Artin's book [1], *Geometric Algebra.* Two sources are particularly influential in Chapters 3 and 4. The first is Pinkall's dissertation [1] and his subsequent paper [4], which have proven to be remarkably fruitful. Secondly, the approach to the study of Legendre submanifolds using the method of moving frames is due to Shiing–Shen Chern, and was first presented in two papers by Chern and myself [1]–[2]. These two papers and indeed this monograph grew out of my work with Professor Chern during my 1985–86 sabbatical at Berkeley. I am very grateful to Professor Chern for many helpful discussions and insights.

I also want to thank several other mathematicians for their personal contributions. Katsumi Nomizu introduced me to Pinkall's work and Lie sphere geometry in 1982, and his seminar at Brown University has been the site of many enlightening discussions on the subject since that time. Thomas Banchoff introduced me to the cyclides of Dupin in the early seventies, when I was a graduate student, and he has provided me with several key insights over the years, particularly through his films. Patrick Ryan has contributed significantly to my understanding of this subject through many lectures and discussions. I also want to acknowledge helpful conversations and correspondence on various aspects of the subject with Steven Buyske, Sheila Carter, Leslie Coghlan, Josef Dorfmeister, Thomas Hawkins, Wu–Yi Hsiang, Nicolaas Kuiper, Martin Magid, Reiko Miyaoka, Ross Niebergall, Tetsuya Ozawa, Richard Palais, Ulrich Pinkall, Helmut Reckziegel, Chuu–Lian Terng, Gudlaugur Thorbergsson, and Alan West.

This book grew out of lectures given in the Brown University Differential Geometry Seminar in 1982–83 and subsequent lectures given to the Clavius Group during the summers of 1985–89 at the University of Notre Dame, the University of California at Berkeley, Fairfield University and the Institute of Advanced Study. I want to thank my fellow members of the Clavius Group for their support of these lectures and many enlightening remarks. I also acknowledge with gratitude the hospitality of the institutions mentioned above.

I wish to thank my colleagues in the Department of Mathematics at the College of the Holy Cross, several of whom are my former teachers, for many insights and much encouragement over the years. I especially wish to mention

my first teacher in linear algebra and real analysis, Leonard Sulski, who recently passed away after a courageous battle against leukemia. Professor Sulski was a superb, dedicated teacher, and a good and generous man. He will be missed by all who knew him.

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Finally, I am most grateful to my wife, Patsy, and my sons, Tom, Mark and Michael, for their patience, understanding and encouragement during this lengthy project.

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