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(continued after index)

Dale R. Durran

Numerical Methods for Wave Equations in Geophysical Fluid Dynamics

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To every hand that's touched the Wall

Series Preface

Mathematics is playing an ever more important role in the physical and biological sciences, provoking a blurring of boundaries between scientific disciplines and a resurgence of interest in the modern as well as the classical techniques of applied mathematics. This renewal of interest, both in research and teaching, has led to the establishment of the series: *Texts in Applied Mathematics (TAM)*.

The development of new courses is a natural consequence of a high level of excitement on the research frontier as newer techniques, such as numerical and symbolic computer systems, dynamical systems, and chaos, mix with and reinforce the traditional methods of applied mathematics. Thus, the purpose of this textbook series is to meet the current and future needs of these advances and encourage the teaching of new courses.

TAM will publish textbooks suitable for use in advanced undergraduate and beginning graduate courses, and will complement the *Applied Mathematical Sciences (AMS)* series, which will focus on advanced textbooks and research level monographs.

Preface

This book is designed to serve as a textbook for graduate students or advanced undergraduates studying numerical methods for the solution of partial differential equations governing wave-like flows. Although the majority of the schemes presented in this text were introduced in either the applied-mathematics or atmospheric-science literature, the focus is not on the nuts-and-bolts details of various atmospheric models but on fundamental numerical methods that have applications in a wide range of scientific and engineering disciplines. The prototype problems considered include tracer transport, shallow-water flow and the evolution of internal waves in a continuously stratified fluid.

A significant fraction of the literature on numerical methods for these problems falls into one of two categories, those books and papers that emphasize theorems and proofs, and those that emphasize numerical experimentation. Given the uncertainty associated with the messy compromises actually required to construct numerical approximations to real-world fluid-dynamics problems, it is difficult to emphasize theorems and proofs without limiting the analysis to classical numerical schemes whose practical application may be rather limited. On the other hand, if one relies primarily on numerical experimentation it is much harder to arrive at conclusions that extend beyond a specific set of test cases. In an attempt to establish a clear link between theory and practice, I have tried to follow a middle course between the theorem-and-proof formalism and the reliance on numerical experimentation. There are no formal proofs in this book, but the mathematical properties of each method are derived in a style familiar to physical scientists. At the same time, numerical examples are included that illustrate these theoretically derived properties and facilitate the intercomparison of various methods.

A general course on numerical methods for geophysical fluid dynamics might draw on portions of the material presented in Chapters 2 through 6. Chapter 2 describes the largely classical theory of finite-difference approximations to the one-way wave equation (or alternatively the constant-wind-speed advection equation). The extension of these results to systems of equations, several space dimensions, dissipative flows and nonlinear problems is discussed in Chapter 3. Chapter 4 introduces series-expansion methods with emphasis on the Fourier and spherical-harmonic spectral methods and the finite-element method. Finite-volume methods are discussed in Chapter 5 with particular attention devoted to methods for simulating the transport of scalar fields containing poorly resolved spatial gradients. Semi-Lagrangian schemes are analyzed in Chapter 6. Both theoretical and applied problems are provided at the end of each chapter. Those problems that require numerical computation are marked by an asterisk.

In addition to the core material in Chapters 2 through 6, the introduction in Chapter 1 discusses the relation between the equations governing wave-like geophysical flows and other types of partial differential equations. Chapter 1 concludes with a short overview of the strategies for numerical approximation that are considered in detail throughout the remainder of the book. Chapter 7 examines schemes for the approximation of slow moving waves in fluids that support physically insignificant fast waves. The emphasis in Chapter 7 is on atmospheric applications in which the slow wave is either an internal gravity wave and the fast waves are sound waves, or the slow wave is a Rossby wave and the fast waves are both gravity waves and sound waves. Chapter 8 examines the formulation of wave-permeable boundary conditions for limited-area models with emphasis on the shallow-water equations in one and two dimensions and on internally stratified flow.

Many numerical methods for the simulation of internally stratified flow require the repeated solution of elliptic equations for pressure or some closely related variable. Due to the limitations of my own expertise and to the availability of other excellent references I have not discussed the solution of elliptic partial differential equations in any detail. A thumbnail sketch of some solution strategies is provided in Section 7.1.3; the reader is referred to Chapter 5 of Ferziger and Perić (1997) for an excellent overview of methods for the solution of elliptic equations arising in computational fluid dynamics.

I have attempted to provide sufficient references to allow the reader to further explore the theory and applications of many of the methods discussed in the text, but the reference list is far from encyclopedic and certainly does not include every worthy paper in the atmospheric science or applied mathematics literature. References to the relevant literature in other disciplines and in foreign language journals is rather less complete.¹

¹ Those not familiar with the atmospheric science literature may be surprised by the number of references to *Monthly Weather Review*, which despite its title, has become the primary American journal for the publication of papers on numerical methods in atmospheric science.

This book would not have been written without the generous assistance of several colleagues. Christopher Bretherton, in particular, provided many perceptive answers to my endless questions. J. Ray Bates, Byron Boville, Michael Cullen, Marcus Grote, Robert Higdon, Randall LeVeque, Christoph Schär, William Skamarock, Piotr Smolarkiewicz, and David Williamson all provided very useful comments on individual chapters. Many students used earlier versions of this manuscript in my courses in the Atmospheric Sciences Department at the University of Washington, and their feedback helped improve the clarity of the manuscript. Two students to whom I am particularly indebted are Craig Epifanio and Donald Slinn. I am also grateful to James Holton for encouraging me to undertake this project.

It is my pleasure to acknowledge the many years of support for my numerical modeling efforts provided by the Mesoscale Dynamic Meteorology Program of the National Science Foundation. Additional support for my atmospheric simulation studies has been provided by the Coastal Meteorology ARI of the Office of Naval Research. Part of this book was completed while I was on sabbatical at the Laboratoire d'Aérodynamique of the Université Paul Sabatier in Toulouse, France, and I thank Daniel Guedalia and Evelyne Richard for helping make that year productive and scientifically stimulating.

As errors in the text are identified, they will be posted on the web at <http://www.atmos.washington.edu/methods.for.waves>, which can be accessed directly or via Springer's home page at <http://www.springer-ny.com>. I would be most grateful to be advised of any typographical or other errors by electronic mail at dale.durran@atmos.washington.edu.

Seattle, Washington

DALE R. DURRAN

Cover art: The three curves plot solutions to the linearized Rossby–adjustment problem. The governing equations and physical parameters for this problem are identical to those given in Problem 12 of Chapter 3, except that the spatial domain is $-400 \text{ km} \leq x \leq 400 \text{ km}$ with open lateral boundaries, and the initial condition for the free-surface displacement is $h(x, t = 0) = \arctan(x/20 \text{ km})$. The curves shown are plots of $u(x, t = 943 \text{ s})$, $u(x, t = 1222 \text{ s})$, and $u(x, t = 1501 \text{ s})$ on an artistically cropped portion of the sub-domain $x > 0$.

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