Time-Varying Systems and Computations

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by

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Contents

Preface	ix
Acknowledgments	xiii
1. INTRODUCTION	1
1.1 Computational linear algebra and time-varying systems	1
1.2 Objectives of computational modeling	7
1.3 Connections and alternative approaches	13
Part I REALIZATION	
2. NOTATION AND PROPERTIES OF NON-UNIFORM SPACES	19
2.1 Spaces of non-uniform dimensions	20
2.2 Shifts and diagonal representations	26
2.3 Notes	30
3. TIME-VARYING STATE SPACE REALIZATIONS	33
3.1 Realizations of a transfer operator	34
3.2 Special classes of time-varying systems	41
3.3 Examples and extensions	46
3.4 Realization theory for finite matrices	52
3.5 Identification from input-output data	62
3.6 Realization theory for matrices of low displacement rank	65
4. DIAGONAL ALGEBRA	73
4.1 Sequences of diagonals	74
4.2 The diagonal algebra of $\mathcal{X}_2$	76
4.3 Sliced bases and projections in $\mathcal{X}_2$	79
5. OPERATOR REALIZATION THEORY	87
5.1 The Hankel operator	88
5.2 Reachability and observability operators	91
5.3 Reachability and observability Gramians	95

5.4	Abstract realization theory	102
5.5	Notes	116
6.	ISOMETRIC AND INNER OPERATORS	121
6.1	Realization of inner operators	122
6.2	External factorization	126
6.3	State-space properties of isometric systems	132
6.4	Beurling-Lax like theorem	136
6.5	Example	142
7.	INNER-OUTER FACTORIZATION AND OPERATOR INVERSION	145
7.1	Introduction	146
7.2	Inner-outer factorizations	149
7.3	Operator inversion	165
7.4	Examples	172
7.5	Zero structure and its limiting behavior	179
7.6	Notes	186
Part II INTERPOLATION AND APPROXIMATION		
8.	J-UNITARY OPERATORS	191
8.1	Scattering operators	191
8.2	Geometry of diagonal J-inner product spaces	200
8.3	State space properties of J-unitary operators	204
8.4	Past and future scattering operators	210
8.5	J-unitary external factorization	215
8.6	J-lossless and J-inner chain scattering operators	218
8.7	The mixed causality case	223
9.	ALGEBRAIC INTERPOLATION	233
9.1	Diagonal evaluations or the W-transform	235
9.2	The algebraic Nevanlinna-Pick problem	237
9.3	The tangential Nevanlinna-Pick problem	242
9.4	The Hermite-Fejer interpolation problem	242
9.5	Conjugation of a left interpolation problem	245
9.6	Two sided interpolation	250
9.7	The four block problem	260
10.	HANKEL-NORM MODEL REDUCTION	263
10.1	Introduction	264
10.2	Approximation via indefinite interpolation	269
10.3	State realization of the approximant	276
10.4	Parametrization of all approximants	282
10.5	Schur-type recursive interpolation	292

10.6 The Nehari problem	300
10.7 Concluding remarks	306
11. LOW-RANK MATRIX APPROXIMATION AND SUBSPACE TRACKING	307
11.1 Introduction	307
11.2 J-unitary matrices	309
11.3 Approximation theory	311
11.4 Hyperbolic QR factorization	316
11.5 Hyperbolic URV decomposition	322
11.6 Updating the SSE-2	325
11.7 Notes	331
 Part III FACTORIZATION	
12. ORTHOGONAL EMBEDDING	337
12.1 Introduction and connections	338
12.2 Strictly contractive systems	343
12.3 Contractive systems: the boundary case	347
12.4 Lossless embedding	351
12.5 Numerical issues	354
12.6 Notes	359
13. SPECTRAL FACTORIZATION	363
13.1 Introduction	363
13.2 Spectral factorization	365
13.3 Computational issues	371
13.4 Convergence of the Riccati recursion	372
13.5 Connections	376
13.6 Notes	380
14. LOSSLESS CASCADE FACTORIZATIONS	383
14.1 Time-invariant cascade factorizations	383
14.2 Parsimonious parametrization of contractive LTI systems	390
14.3 Time-varying Σ -based cascade factorization	397
14.4 Time-varying Θ -based cascade factorization	410
15. CONCLUSION	419
Appendices	423
A– Hilbert space definitions and properties	423
References	435
Glossary of notation	453
Index	455

Complex function theory and linear algebra provide much of the basic mathematics needed by engineers engaged in numerical computations, signal processing or control. The transfer function of a linear time invariant system is a function of the complex variable s or z and it is analytic in a large part of the complex plane. Many important properties of the system for which it is a transfer function are related to its analytic properties. On the other hand, engineers often encounter small and large matrices which describe (linear) maps between physically important quantities. In both cases similar mathematical and computational problems occur: operators, be they transfer functions or matrices, have to be simplified, approximated, decomposed and realized. Each field has developed theory and techniques to solve the main common problems encountered.

Yet, there is a large, mysterious gap between complex function theory and numerical linear algebra. For example, complex function theory has solved the problem to find analytic functions of minimal complexity and minimal supremum norm that approximate given values at strategic points in the complex plane. They serve *e.g.*, as optimal approximants for a desired behavior of a system to be designed. No similar approximation theory for matrices existed until recently, except for the case where the matrix is (very) close to singular. The relevant approximation theory in the complex plane is spectacular and has found a manifold of applications such as broadband matching, minimal sensitivity control, and the solution of differential game problems. A similar “linear algebra” result would without doubt be very desirable. Over the years we have discovered that a strong link between the two theories can indeed be developed.

To establish this link, one has to move away from the classical idiosyncrasies of the two basic theories, and develop a new and somewhat unusual paradigm, which, however, turns out to be quite natural and practical once one gets used to it. Classical matrix theory and linear algebra act on vectors and matrices. Very early in the development of these theories it was found beneficial to move from single scalar quantities and variables to vector representations. This has been an important lift in the level of abstraction, with great importance for physics and engineering, and also largely motivated by them. It has allowed for compact, insightful algebraic notations which have been adopted by a large variety of fields in which multidimensional objects interact with each other. Mechanics, electromagnetism, quantum mechanics, operations re-

search, electrical network theory and signal processing all are fields which have been deeply influenced by vector and matrix calculus.

With the advent of powerful electronic computing, global vector or matrix-vector operations may even be viewed as atomic numerical operations. A vector computer can be programmed to execute them in parallel by a single instruction. A matrix-vector or matrix-matrix multiplication, a matrix inversion, and more complicated operations such as the calculation of matrix eigenvalues, can easily be conceived as simple sequences of such massive vector operations. In this book we will add another basic, vector-like quantity to the arsenal of objects handled by linear algebra. The new object represents a diagonal of a matrix or an operator. Thus, in addition to matrix operations acting on rows or columns, we shall consider elementary operations on diagonals. These, in fact, can be handled with the same ease by a vector or parallel computer, but they will have a very different algebraic interpretation.

What is to be gained by such an approach? In the course of the book, we develop a forceful argument that indeed, it allows us to solve several problems in linear algebra whose solutions were either unknown, or obscured by the traditional approach. In addition, the theory also creates its own class of new problems. The missing theoretical link is provided by system theory, in our case the theory of linear, time discrete and time varying dynamical systems. For example, we look at the meaning of “computational complexity” from a system theoretical point of view. Up to now, classical linear algebra had only a limited notion of complexity, restricted to either matrices that are sparse (most entries equal to zero), or matrices that are close to singular. The sparse structure is easily destroyed by algebraic operations: even the inverse of such a matrix is not sparse, and as a result, it seems that multiplication by this inverse is a full-complexity operation. This does not happen with a system theoretic “realization”: it is straightforward to show that a minimal realization of the inverse has the same complexity as one for the original. In addition, system theory will allow us to derive a powerful approximation theory that maps a given matrix to a matrix of lowest computational complexity (in the system theoretical sense), given a certain tolerance.

System theory has already had a significant impact on linear algebra, mostly in the study of Toeplitz matrices and similar structured matrix problems. These are connected to time-invariant systems. Our approach in this book is complementary: we generalize to *time-varying* systems, which allows to describe any matrix. The structure in the matrix we are looking for is now less obvious, it is connected to the rank of certain strategic submatrices. In the course of the book, several classical results in the theory of time-varying systems are recovered: *e.g.*, we hit on the all-pervasive time-varying Riccati equation, the bounded real lemma and the related Kalman-Yakubovitch-Popov lemma. Still, we believe that we are placing the theory in the context of a new “paradigm”, *i.e.*, a realm of problems and solutions with their own dynamics. Indeed, several results in time-varying system theory that were only known as abstract theory (such as proofs by Arveson of the existence of inner-outer factorizations) have now become explicit “constructive operator theory”. Significant new results in this context are the time-varying Hankel-norm approximation theory, as well as the solution of several interpolation problems, leading to a generalization of the minimal sensitivity problem and optimal control theory.

An added value of the book is the very explicit link which it lays between numerical linear algebra and generalizations of analytic function theory on the open unit disc, as traditionally applied to transfer function calculus. The reader will discover the algebraic generalizations of a host of classical interpolation problems: Schur, Nevanlinna-Pick, Caratheodory-Fejer, Nehari, Schur-Takagi. These provide natural solutions to nice problems in algebra. Conversely, elementary methods in numerical analysis have interesting counterparts in system theory where their interpretation is non-trivial. E.g., we show that inner-outer factorization can be viewed as a generalization of QR factorization, and Hankel-norm model reduction can be used for efficient subspace estimation and tracking.

We do not limit ourselves to finite matrices. The connection to system theory allows to derive meaningful results also for “infinite” matrices, or operators on a Hilbert space. From a linear algebra point of view, the results are perhaps uncanny: e.g., the inverse of an upper triangular infinite matrix need not be upper triangular! The connection to time-varying systems gives insight into the mechanics of this: the inverse of an upper operator is upper if and only if the original system has a property which is called “outer”. Even for linear algebra, infinite linear systems are useful: such systems occur e.g., in the discretization of differential equations where the boundary condition is conveniently placed at infinity. Because the matrix entries become constant as we move away from the center of the matrix, it can still be described by a finite number of parameters. It has been amply demonstrated that such a procedure may lead to more accurate overall results. The downside of this generality is perhaps that, in order to obtain precise results, Hilbert space theory plays a major but sometimes also a mere technical role. (We summarize some basic notions of Hilbert space theory in an appendix.)

For whom is this book intended? We suggest the following.

- It can be used as a graduate course in linear time-varying system theory: all the main concepts and problems are there, and they are treated in a direct and streamlined manner. For this purpose we have been somewhat lengthy in developing the basic framework in the first chapters — our excuses to interested mathematicians!
- It can be used as a source of new problems in numerical linear algebra, with a concurrent new methodology to solve them. Several theories in the book scream for in-depth analysis, in particular the theory of optimal sensitivity reduction, the inversion theory for infinite systems of equations and the optimal matrix-operator approximation theory.
- It can also be used as an introductory course in a new topic: “theory of computational systems”. Although the material presented here falls short of a comprehensive theory — the subject matter presently does not go far beyond linear problems and computations — we do think that there is already sufficient information to justify independent interest.

It is our hope that the algebraic system’s community will find inspiration and motivation in the theory presented here. Although it has definite affinities to Arveson’s “Nested Algebras” and Saks and Feintuch’s “Resolution Spaces”, it does have a new flavor, mainly because of its direct link to numerical procedures via the treatment of diagonals as the new vectorial object.

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