Probability and Its Applications

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Intersections of Random Walks

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Preface

A more accurate title for this book would be "Problems dealing with the non-intersection of paths of random walks." These include: harmonic measure, which can be considered as a problem of nonintersection of a random walk with a fixed set; the probability that the paths of independent random walks do not intersect; and self-avoiding walks, i.e., random walks which have no self-intersections. The prerequisite is a standard measure theoretic course in probability including martingales and Brownian motion.

The first chapter develops the facts about simple random walk that will be needed. The discussion is self-contained although some previous exposure to random walks would be helpful. Many of the results are standard, and I have made borrowed from a number of sources, especially the excellent book of Spitzer [65]. For the sake of simplicity I have restricted the discussion to simple random walk. Of course, many of the results hold equally well for more general walks. For example, the local central limit theorem can be proved for any random walk whose increments have mean zero and finite variance. Some of the later results, especially in Section 1.7, have not been proved for very general classes of walks. The proofs here rely heavily on the fact that the increments of simple random walk are bounded and symmetric. While the proofs could be easily adapted for other random walks with bounded and symmetric increments, it is not clear how to extend them to more general walks. Some progress in this direction has been made in [59].

The proof of the local central limit theorem in Section 1.2 follows closely the proof in [65]. The next sections develop the usual probabilistic tools for analyzing walks: stopping times, the strong Markov property, martingales derived from random walks, and boundary value problems for discrete harmonic functions. Again, all of this material is standard. The asymptotics of the Green's function for $d \geq 3$ and of the potential kernel for $d = 2$ are then derived. There is care in these sections in being explicit about the size of the error in asymptotic results. While this makes it a little harder to read initially, it is hoped that this will allow the chapter to be a reference for "well known" facts about simple random walks. The results in the last section of this chapter are analogous to results which are standard in partial differential equations: difference estimates for harmonic functions and Harnack inequality. Unfortunately the discrete versions of these useful results do not seem to be familiar to many people working in random walks. A version of Theorem 1.7.1(a) was first proved in [8]. A number of "exercises" are included in Chapter 1 and the beginning of Chapter 2. It is suggested that the reader do the exercises, and I have felt free to quote results from the exercises later in the book.

Harmonic measure is the subject of the second chapter. By harmonic measure here we mean harmonic measure from infinity, i.e., the hitting distribution of a set from a random walker starting at infinity. There are many ways to show the existence of harmonic measure, see e.g. [65]. Here the existence is derived as a consequence of the results in Section 1.7. This method has the advantage that it gives a bound on the rate of convergence. In Sections 2.2 and 2.3, the idea of discrete capacity is developed. The results of these sections are well known although some of the proofs are new. I take the viewpoint here that capacity is a measure of the probability that a random walk will hit a set. In the process, I completely ignore the interpretation in terms of electrical capacity or equilibrium potentials. Computing harmonic measure or escape probabilities can be very difficult. Section 2.4 studies the example of a line or a line segment and in the process develops some useful techniques for estimating harmonic measure. First, there is a discussion of Tauberian theorems which are used to relate random walks with geometric killing times with random walks with a fixed number of steps (analytically, this is a comparison of a sequence and its generating function). Then the harmonic measure of a line and a line segment are derived. The earlier estimates are standard. The estimate for the endpoint of a line segment in two dimensions (2.41) was first derived by Kesten [35] using a different argument. The argument here which works for two and three dimensions first appeared in [45]. The next section gives upper bounds for harmonic measure. The bound in terms of the cardinality of the set has been known for a long time. The bound for connected sets in terms of the radius is a discrete analogue of the Beurling projection theorem (see [1]) and was first proved for $d = 2$ by Kesten [35]. The three dimensional result is new here; however, the proofs closely follow those in [35]. The final section gives an introduction to diffusion limited aggregation (DLA), a growth model first introduced by Witten and Sander [73]. The bounds from the previous section are used to give bounds on the growth rate of DLA clusters; again, the result for $d = 2$ was first proved by Kesten [36] and the three dimensional result uses a similar proof.

The next three chapters study the problem of intersections of random walks or, more precisely, the probability that the paths of independent random walks intersect. We will not discuss in detail what the typical intersection set looks like. This has been studied by a number of authors under the name "intersection local time", see e.g. [47] . The discussion on the probability of intersection follows the results in [11,12,39,40,41,45]. Chapter 3 sets the basic framework and proves some of the easier results. In Section 3.2, the expected number of intersections is calculated (a straightforward computation) and one lower bound on the hitting probability is given, using a proof adapted from [22]. The expected number of intersections gives a natural conjecture about the order of the probability of "long-range" intersections. This conjecture is proved in the next two sections. For $d \neq 4$, the proof requires little more than the estimate of two moments of the number of intersections. More work is needed in the critical dimension $d = 4$; the proof we give in Section 3.4 uses the properties of a certain random variable which has a small variance in four dimensions. This random variable is used in the next chapter when more precise estimates are given for $d = 4$. The problem of estimating the probability of intersections of two random walks starting at the same point is then considered. It turns out that the easier problem to discuss is the probability that a "two-sided" walk does not intersect a "one-sided" walk. The probability of no intersection in this case is shown to be equal to the inverse of the expected number of intersections, at least up to a multiplicative constant. This fact is proved in Sections 3.5, 3.6, and 4.2. This then gives some upper and lower bounds for the probability that two one-sided walks starting at the same point do not intersect. The material in this chapter essentially follows the arguments in [39,45]. Some of these results have been obtained by other means [2,23,58], and some simplifications from those papers are reflected in the treatment here .

The techniques of Chapter 3 are not powerful enough to analyze the probability that two one-sided walks starting at the origin do not intersect. There are a number of reasons to be interested in this problem. It is a random walk analogue of a quantity that arises in a number of problems in mathematical physics (e.g., a similar quantity arises in the discussion of critical exponents for self-avoiding walks in Section 6.3). Also, some of the techniques used in nonrigorous calculations in mathematical physics can be applied to this problem, see e.g. [16,17], so rigorous analysis of this problem can be used as a test of the effectiveness of these nonrigorous methods. Unfortunately, there is not yet a complete solution to this problem; Chapters 4 and 5 discuss what can be proved.

In four dimensions, the probability of nonintersection goes to zero like an inverse power of the logarithm of the number of steps. The techniques of Chapter 3 give bounds on this power; in Chapter 4, the exact power is derived. The first part of the derivation is to give asymptotic expressions for the probability of "long-range" intersections (the results of the previous chapter only give expressions up to a multiplicative constant). Sections 4.3 and 4.4 derive the expressions, using a natural relationship between long-range intersections and intersections of a two-sided walk with a one-sided walk. The next section derives the exact power of the logarithm. It essentially combines the result on long-range intersection with an estimate on asymptotic independence of short-range and long-range intersections to estimate the "derivative" of the probability of no intersection. The final section discusses a similar problem, the mutual intersections of three walks in three dimensions. The results are analogous to those of two walks in four dimensions. Some of these results appeared in [41]. One new result is Theorem 4.5.4, which gives the exact power of the logarithm for the probability of no intersection.

The next chapter considers the intersection probability in dimensions two and three. Here the probability of no intersection goes to zero like a power of the number of steps. Again, the results of Chapter 3 can be used to give upper and lower bounds for the exponent. The first thing that is proved is that the exponent exists. This is done in Sections 5.2 and 5.3 by relating it to an exponent for intersections of paths of Brownian motions. Some estimates are derived for the exponent in the remainder of the chapter. First a variational formulation of the exponent is given. The formulation is in terms of a function of Brownian motion. Bounds on this function then give bounds on the exponent. Section 5.5 gives a lower bound for the intersection exponent in two dimensions by comparing it to a different exponent which measures the probability that a Brownian motion makes a closed loop around the origin. The last section gives an upper bound in two and three dimensions.

The last two chapters are devoted to self-avoiding walks, i.e., random walks conditioned to have no (or few) self-intersections. Sections 6.2 and 6.3 discuss the usual (strictly) self-avoiding walk, i.e., simple random walk of a given length with no self-intersections. The connective constant is defined, and then there is a discussion of the critical exponents for the model. The critical exponents are discussed from a probabilistic viewpoint; however, the discussion is almost entirely heuristic. The few nontrivial results about the self-avoiding walk have been obtained from either combinatorial or (mathematical physics) field-theoretic arguments. We mention a few of these results here. There is a forthcoming book by N. Madras and G. Slade in this series which will cover these topics in more detail. The next two sections discuss other models for self-avoiding or self-repelling walks. They fall neatly into two categories: configurational models (Section 6.4) and kinetically growing walks (Section 6.5). The final section gives a brief introduction to the problem of producing self-avoiding walks on the computer, a topic which has raised a number of interesting mathematical questions.

The last chapter discusses a particular model for self-avoiding walks, the loop-erased or Laplacian random walk. This model can be defined in two equivalent ways, one by erasing loops from the paths of simple random walk and the other as a kinetically growing walks with steps taken weighted according to harmonic measure. This model is similar to the usual self-avoiding walk in a number of ways: the critical dimension is four; there is convergence to Brownian motion for dimensions greater than or equal to four, with a logarithmic correction in four dimensions; nontrivial exponents describe the mean-square displacement below four dimensions. Unfortunately, this walk is not in the same universality class as the usual self-avoiding walk; in particular, the mean-square displacement exponent is different. The basic construction of the process is done in the first four sections. There are some technical difficulties in defining the walk in two dimensions because of the recurrence of simple random walk. These are discussed in Section 7.4. In the next section, estimates on the average amount erased are made. These are then used in Section 7.6 to show that the mean-square displacement exponents are at least as large as the Flory exponents for usual self-avoiding walk. The convergence to Brownian motion in high dimensions is done in the last section. Essentially the result follows from a weak law that says that the amount erased is uniform on each path. The proof follows [38,42]; however, unlike those papers the treatment in this book does not use any nonstandard analysis.

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Notation

We use c, c_1, c_2 to denote arbitrary positive constants, depending only on dimension, which may change from line to line. If a constant is to depend on some other quantity, this will be made explicit. For example, if c depends on α, we write $c(\alpha)$ or c_α. If $g(x)$,$h(x)$ are functions we write $g \sim h$ if they are asymptotic, i.e,

$$\lim_{x \to \infty} \frac{h(x)}{g(x)} = 1.$$

We write $g \asymp h$ if there exist constants c_1, c_2 such that

$$c_1 g(x) \leq h(x) \leq c_2 g(x).$$

Finally we write $g \approx h$ if $\ln g \sim \ln h$.

We write $h(x) = O(g(x))$ if $h(x) \leq cg(x)$ for some constant c. Again, the implicit assumption is that the constant c depends only on dimension. If we wish to imply that the constant may depend on another quantity, say α, we write $O_\alpha(g(x))$. For example, $\alpha x = O_\alpha(x)$, but it is not true that $\alpha x = O(x)$. Similarly, we write $h(x) = o(g(x))$ if $h(x)/g(x) \to 0$. By implication, the rate of convergence depends on no other parameters, except dimension. We will write o_α to indicate a dependence on the parameter α.

Similar conventions hold for limits as $x \to 0$ or $x \to 1-$.

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