Graduate Texts in Mathematics 103

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7 SERRE. A Course in Arithmetic.
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46 LOÈVE. Probability Theory II. 4th ed.
47 MOISE. Geometric Topology in Dimensions 2 and 3.

continued after Index

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With 132 Illustrations

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Foreword

The present book is meant as a text for a course on complex analysis at the advanced undergraduate level, or first-year graduate level. Somewhat more material has been included than can be covered at leisure in one term, to give opportunities for the instructor to exercise his taste, and lead the course in whatever direction strikes his fancy at the time. A large number of routine exercises are included for the more standard portions, and a few harder exercises of striking theoretical interest are also included, but may be omitted in courses addressed to less advanced students.

In some sense, I think the classical German prewar texts were the best (Hurwitz–Courant, Knopp, Bieberbach, etc.) and I would recommend to anyone to look through them. More recent texts have emphasized connections with real analysis, which is important, but at the cost of exhibiting succinctly and clearly what is peculiar about complex analysis: the power series expansion, the uniqueness of analytic continuation, and the calculus of residues. The systematic elementary development of formal and convergent power series was standard fare in the German texts, but only Cartan, in the more recent books, includes this material, which I think is quite essential, e.g., for differential equations. I have written a short text, exhibiting these features, making it applicable to a wide variety of tastes.

The book essentially decomposes into two parts.

The *first part*, Chapters I through VIII, includes the basic properties of analytic functions, essentially what cannot be left out of, say, a one-semester course.

I have no fixed idea about the manner in which Cauchy's theorem is to be treated. In less advanced classes, or if time is lacking, the usual hand waving about simple closed curves and interiors is not entirely inappropriate. Perhaps better would be to state precisely the homological version and omit the formal proof. For those who want a more thorough understanding, I include the relevant material.

Artin originally had the idea of basing the homology needed for complex variables on the winding number. I have included his proof for Cauchy's theorem, extracting, however, a purely topological lemma of independent interest, not made explicit in Artin's original *Notre Dame* notes (cf. collected works) or in Ahlfor's book closely following Artin. I have also included the more recent proof by Dixon, which uses the winding number, but replaces the topological lemma by greater use of elementary properties of analytic functions which can be derived directly from the local theorem. The two aspects, homotopy and homology, both enter in an essential fashion for different applications of analytic functions, and neither is slighted at the expense of the other.

Most expositions usually include some of the global geometric properties of analytic maps at an early stage. I chose to make the preliminaries on complex functions as short as possible to get quickly into the analytic part of complex function theory: power series expansions and Cauchy's theorem. The advantages of doing this, reaching the heart of the subject rapidly, are obvious. The cost is that certain elementary global geometric considerations are thus omitted from Chapter I, for instance, to reappear later in connection with analytic isomorphisms (Conformal Mappings, Chapter VII) and potential theory (Harmonic Functions, Chapter VIII). I think it is best for the coherence of the book to have covered in one sweep the basic analytic material before dealing with these more geometric global topics. Since the proof of the general Riemann mapping theorem is somewhat more difficult than the study of the specific cases considered in Chapter VII, it has been postponed to the second part.

The *second part* of the book, Chapters IX through XIV, deals with further assorted analytic aspects of functions in many directions, which may lead to many other branches of analysis. I have emphasized the possibility of defining analytic functions by an integral involving a parameter and differentiating under the integral sign. Some classical functions are given to work out as exercises, but the gamma function is worked out in detail in the text, as a prototype. *The chapters in this part are essentially logically independent and can be covered in any order, or omitted at will.*

In particular, the chapter on analytic continuation, including the Schwarz reflection principle, and/or the proof of the Riemann mapping theorem could be done right after Chapter VII, and still achieve great coherence.

As most of this part is somewhat harder than the first part, it can easily be omitted from a course addressed to undergraduates. In the same spirit, some of the harder exercises in the first part have been starred, to make their omission easy.

In this second edition, I have rewritten many sections, and I have added some material. I have also made a number of corrections whose need was pointed out to me by several people. I thank them all.

I am much indebted to Barnet M. Weinstock for his help in correcting the proofs, and for useful suggestions.

SERGE LANG

Prerequisites

We assume that the reader has had two years of calculus, and has some acquaintance with epsilon–delta techniques. For convenience, we have recalled all the necessary lemmas we need for continuous functions on compact sets in the plane.

We use what is now standard terminology. A function

$$f: S \to T$$

is called **injective** if $x \neq y$ in S implies $f(x) \neq f(y)$. It is called **surjective** if for every z in T there exists $x \in S$ such that $f(x) = z$. If f is surjective, then we also say that f maps S **onto** T. If f is both injective and surjective then we say that f is **bijective**.

Given two functions f, g defined on a set of real numbers containing arbitrarily large numbers, and such that $g(x) \geqq 0$, we write

$$f \ll g \qquad \text{or} \qquad f(x) \ll g(x) \qquad \text{for} \quad x \to \infty$$

to mean that there exists a number $C > 0$ such that for all x sufficiently large, we have

$$|f(x)| \leqq Cg(x).$$

Similarly, if the functions are defined for x near 0, we use the same symbol $\ll$ for $x \to 0$ to mean that there

$$|f(x)| \leqq Cg(x)$$

for all x sufficiently small (there exists $\delta > 0$ such that if $|x| < \delta$ then $|f(x)| \leqq Cg(x)$). Often this relation is also expressed by writing

$$f(x) = O(g(x)),$$

which is read: $f(x)$ is **big oh of** $g(x)$, for $x \to \infty$ or $x \to 0$ as the case may be.

We use $]a, b[$ to denote the **open** interval of numbers

$$a < x < b.$$

Similarly, $[a, b[$ denotes the half-open interval, etc.

Contents