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Yuri Kifer

**Ergodic Theory of
Random Transformations**

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To my family

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Frequently used notations

$\mathcal{B}(M)$ - the Borel σ -algebra of M .

$\mathfrak{C}(M, N)$ -the space of continuous maps from M to N .

\mathfrak{C}^k -class - continuous together with k -derivatives.

Df the differential of a map f .

$\mathbb{E}r$ -the expectation of a random variable r .

\mathfrak{F} -a space of transformations on \mathbf{M} .

f - a random transformation with a distribution \mathfrak{m} .

F - a random bundle map with a distribution \mathfrak{n} .

${}^n f = f_n \circ \dots \circ f_1, \quad {}^n F = F_n \circ \dots \circ F_1, \quad D {}^n f$ means the differential of ${}^n f$.

$h_\rho(f)$ - the metric entropy of f with respect to an invariant measure ρ .

$h(f)$ - the topological entropy of f .

\mathbb{I} - the unit interval.

$\mathfrak{L}(M, \eta)$ - the space of functions g with $\int_M |g| d\eta < \infty$
 $\mathfrak{p} = \mathfrak{m}^\infty$ or $\mathfrak{p} = \mathfrak{n}^\infty$.

$\mathbb{P}\{A\}$ - the probability of A .

χ_A - the indicator of a set A i.e., $\chi_A(x) = 1$ if $x \in A$ and $= 0$ for otherwise.

$\mathcal{P}(M)$ - the space of probability measures on M .

Π^{m-1} - the $(m-1)$ -dimensional projective space.

\mathbb{R}^m - the m -dimensional Euclidean space.

\mathbb{S} - the unit circle.

\mathcal{T} -a space of vector bundle maps.

TM - the tangent bundle of a smooth manifold M .

$$\Omega = \mathcal{T}^\infty \text{ or } \Omega = \mathcal{T}^\infty.$$

▪ - the end of the proof.

Statement i,j - i denotes the section and j denotes the number of this statement in the section. The Roman number at the beginning (for instance, III.1.2) means the number of the chapter.