PART B

RANDOM SCHRÖDINGER OPERATORS

INTRODUCTION

Numerous equations arising from one dimensional discrete physical systems lead to the analysis of a linear second order difference operator H, acting on a complex sequence Ψ_n , $n \in \mathbb{Z}$, by :

$$(H\Psi)_n = b_n^{-1} \left((-\Delta\Psi)_n + a_n \Psi_n\right)$$

In this formula, Δ is the discrete Laplacian $(\Delta\Psi)_n = \Psi_{n+1} + \Psi_{n-1} - 2\Psi_n$ and a_n, b_n , are two fixed sequences of real numbers with $b_n > 0$, representing the physical properties of the medium. Generally such operators are associated to "time dependent" equations and we give some typical examples :

(i) A solution of the Schrödinger equation $i \frac{\partial\phi}{\partial t} = H\phi$ of the form $\phi(n,t) = \Psi_n e^{-i\lambda t}$ satisfies $H\Psi = \lambda\Psi$ where H is the classical Schrödinger operator, that is the operator associated to $b_n = 1$, $\forall n \in \mathbb{Z}$, and a_n is the potential at site n.

(ii) A solution of the wave equation $b_n \frac{\partial^2\phi}{\partial t^2} = (\Delta\phi)_n$ of the form $\phi(n,t) = \Psi_n e^{i\lambda t}$ satisfies $H\Psi = \lambda^2\Psi$ where H is the "Helmotz operator" that is the operator associated to $a_n = 0$, $\forall n \in \mathbb{Z}$, and b_n is the diffusion coefficient at site n.

(iii) A solution of the heat equation $b_n \frac{\partial\phi}{\partial t} = (\Delta\phi)_n$ of the form $\phi(n,t) = \Psi_n e^{-\lambda t}$ satisfies $H\Psi = \lambda\Psi$ where H is the Helmotz operator.

Similar equations and operators also appear in quasi-one dimensional systems associated to an infinite wire of finite cross section with ℓ sites. We have only to replace the sites n by (i,n) where $i \in (1, \ldots, \ell)$, (the integer ℓ is called the width of the strip) and the real sequences a_n, b_n by matrices sequences.

It is well known that the spectral properties of the operator H viewed as a self adjoint operator on an Hilbert space, govern the asymptotic behavior of the solutions of the associated time dependent equation. In quantum mechanics the number $|\phi(t,n)|^2$ (normalized in a suitable way) associated to a solution $\phi(t,n) = (e^{-itH}\Psi)_n$ of the time dependent Schrödinger equation, represents the probability of presence of a particle at the site n at time t. Roughly speaking, when Ψ is associated to the continuous spectrum of H then we have the "diffusion" behavior $\lim_{T\to+\infty} \frac{1}{2T}\int_{-T}^{T} |\phi(t,n)|^2 dt = 0$, and if Ψ is associated to the point spectrum of H then we have the "localization" property $\lim_{N\to+\infty} \operatorname{Sup}_t \sum_{|n|\geq N} |\phi(t,n)|^2 = 0$. (See D. Ruelle [53] for more details). In deterministic systemswith periodic structure, it is known that only diffusion behavior occurs. But it has been remarked that localization appears when this regular structure is perturbed by impurities or inhomogeneity in the medium. Thus a "metallic" wire suddenly becomes an insulator. We first give below a brief historical survey of the mathematical approaches to this subject.

It was first announced by P.W. Anderson [1] (1958) that for the classical multi-dimensional Schrödinger operator with an independent identically distributed family of random potentials, the spectrum has to be pure point for a "typical sample" assuming the disorder "large enough". It was later conjectured by N. Mott and W.D. Twose [46] (1961) that in the one dimensional case, this should be true at any disorder. The works of H. Furstenberg, H. Kesten [20] (1960), H. Furstenberg [19] (1963), V. Osseledec [49] (1968) provided the essential mathematical background used in the first rigorous approaches of the subject. It was first proved by H. Matsuda, K. Ishii [44] (1970), A. Casher J.L. Lebowitz [10] (1971), L.A. Pastur [50] (1973), Y. Yoshioka [64] (1973) that there does not exist an absolutely continuous component in the spectrum of H. An essential step was achieved in 1973 when I. Ja. Goldsheid, S.A. Molcanov, and L.A. Pastur [24] gave the first proof to the conjecture of Mott and Twose (they actually dealed with the continuous case). Their original proof has been later considerably simplified and extended by R. Carmona [7] (1982), [8] (1983), G. Royer [52] (1983), J. Brossard (1983). In the "discrete" case the same result has been obtained by H. Kunz, B. Souillard [37] (1980), J. Lacroix [38] (1982), F. Delyon, H. Kunz, B. Souillard [13] (1983). Moreover Goldsheid gave a similar announcement in a strip [23] (1981) and the proof can be found in J. Lacroix [39] (1983) [40] [41] (1984). All these previous proofs of localization are rather technical and at times the essential guiding principles are not easily understood. Fortunately, in the late of 1984, S. Kotani [36] clarified the situation, giving a rigorous statement to an earlier claim of R.E. Borland [5] (1963).

Our essential goal is to provide a direct and unified treatment to the foregoing problems, in the general setting of operators H introduced at the begining of this discussion here in after called Schrödinger operators. The essential tool will be the theory of products of i.i.d. random matrices developed in the first part of this book. We are mainly concerned with the independent case but a lot of definitions and properties are given in the general ergodic case. The "almost periodic case" is also of great physical and theoretical interest but most of the proofs have nothing to do with random matrices. Interested readers have to look at the survey of B. Simon [57] where they can also find an extensive bibliography. Since it seems that the theory of random matrices can hardly be used in the multidimensional case (up to now,limiting procedures in strips whose width goes to infinity have not been successful) we restrict ourselves to the one dimensional case and strips.

In chapter I the essential definitions and properties related to the spectral analysis of the deterministic operator H are given. In particular we construct a sequence of "good approximations" of the spectral measure of H and establish the existence of "slowly" increasing generalized eigenfunctions. Moreover the links between the singularity of the spectrum and the fundamental notion of hyperbolic behavior of a product of matrices are pointed out.

In chapter II we define an ergodic family of Schrödinger operators which contains as essential examples the classical Schrödinger operator and the Helmotz operator. Some weak properties of the spectrum of H considered as a subset of $\mathbb{R}$ are given, before introducing the essential concept of Lyapunov exponent. Positivity of this exponent is carefuly studied since this property is crucial in order to obtain absence of absolutely continuous spectrum. The distribution of states describing the asymptotic behavior of the eigenvalues distribution for the operator restricted to "boxes" is of great physical interest and we discuss in detail its regularity properties together with the links with the Lyapunov exponent (Thouless formula). Kotani's criterion insuring localization property is then introduced in the general ergodic case but its main application to the independent case is discussed in the following chapter. Finaly we give a straightforward application of the central limit theorem on $SL(2,\mathbb{R})$ to the asymptotic behavior of the conductance.

Chapter III is devoted to the proof of the conjecture of Mott and Twose both in classical Schrödinger and Helmotz case. In the first model, Kotani's criterion gives immediately the solution, but in the general case the proof is more involved and requires some Laplace analysis on $SL(2,\mathbb{R})$. As a consequence extra properties of the distribution of states are obtained.

All these foregoing results are generalized in the chapter IV to the case of a strip. Most of the previous proofs in the one dimensional case can be translated with some care. But some problems are much more involved, especialy positivity of Lyapunov exponents. General results given in the first part of the book are then very useful. The proof of localization in the general case requires also much more work since Laplace analysis on symplectic groups needs some knowledge about symplectic geometry.

Numerous related topics, non-stationary processes for instance, are not tackled when they don't appear as direct applications of products of i.i.d. matrices, thus we don't intend to provide a complete survey in the theory of random Schrödinger operators. Moreover we are aware of that a lot of pioneer and connected works are not cited since we have focused our attention to a precise mathematical aspect of the subject.