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**Products of
Random Matrices with
Applications to
Schrödinger Operators**

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PREFACE

This book presents two closely related series of lectures.

Part A, due to P. Bougerol, is an introduction to the works of Furstenberg, Guivarc'h, Le Page and Raugi on products of random matrices. Only invertible independent identically distributed random matrices satisfying an irreducibility condition are considered. The purpose is to prove in detail the analogues of the classical limit theorem (e.g. law of large numbers, central limit theorem). This part is based on a course given at the University of Paris 7 in 1983.

Part B, due to J. Lacroix, deals with the spectral theory of random Schrödinger operators, where the products of random matrices play a crucial role. It presents a rigorous and unified treatment of the main known results in the one-dimensional discrete case. Since we are aware that some readers are mainly interested in Schrödinger operators, any notion or result needed from part A is clearly restated (but of course not proved again !).

The book is self-contained and should be accessible to readers with minimal background. Since we feel that these topics deserve a large audience we have "tried" to write it English. It is not sure that we have succeeded and we beg the indulgence of the native speaker.

It is our pleasure to thank Yves Guivarc'h, François Ledrappier, Emile Le Page, Albert Raugi, José de Sam Lazaro and Bernard Souillard for enlightening conversations concerning the material presented here and for their encouragements.

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