PART A

LIMIT THEOREMS FOR PRODUCTS OF RANDOM MATRICES

INTRODUCTION

This part is devoted to limit theorems for products of i.i.d. invertible random matrices. The subject matter, initiated by Bellman, was fully developed by Furstenberg, Guivarc'h, Kesten, Le Page and Raugi. This text is intented to serve as an introduction to their work.

We have chosen to keep the level as elementary as possible. This had sometimes led us to write lengthy proofs when shorter ones are available and to omit some important topics. On the other hand the text is self-contained and should be accessible to readers familiar with probability theory as usually developed at graduate level. In particular, no prior knowledge of group theory is assumed.

Let us roughly describe our general line of approach.

We consider a sequence $Y_1, Y_2, \ldots$ of invertible random matrices of order d which are independent and identically distributed. We set $S_o = Id$, $S_1 = Y_1, \ldots, S_n = Y_n \ldots Y_2 Y_1$.

Our purpose is to prove that under an irreducibility condition, the sequence

$$\operatorname{Log} \|S_n x\| \ ; \quad n = 1, 2, \ldots$$

satisfies, for any $x \neq 0$ in $\mathbb{R}^d$, analogues of the classical limit theorems for sums of i.i.d. random variables (e.g. law of large numbers, central limit theorem, ...).

The strategy we shall adopt is based on the following observation.

For any $x \neq 0$ in $\mathbb{R}^d$, let $\bar{x}$ denote its direction (i.e. the class of x in the projective space $P(\mathbb{R}^d)$). Consider the function $F : Gl(d, \mathbb{R}) \times P(\mathbb{R}^d) \longrightarrow \mathbb{R}$ defined by

$$F(Y,\overline{x}) = \mathrm{Log}\,\frac{\|Yx\|}{\|x\|} \quad .$$

Then

$$\mathrm{Log}\,\frac{\|S_n x\|}{\|x\|} = \sum_{i=o}^{n-1} \mathrm{Log}\,\frac{\|S_{i+1}x\|}{\|S_i x\|}$$

$$= \sum_{i=o}^{n-1} \mathrm{Log}\,\left\|Y_{i+1}\frac{S_i x}{\|S_i x\|}\right\|$$

$$= \sum_{i=o}^{n-1} F(Y_{i+1}, \overline{S_i x}) \ .$$

Therefore $\mathrm{Log}\,\|S_n x\|$ can be written as a simple functional of the Markov chain $(Y_{i+1}, \overline{S_i x})$ on $Gl(d,\mathbb{R}) \times P(\mathbb{R}^d)$. Taking into account our irreducibility assumption, the first conclusion we can draw is the law of large numbers. Namely, there exists a real γ_1 , called the upper Lyapunov exponent, such that for any $x \neq 0$

$$\lim_{n\to\infty} \frac{1}{n}\,\mathrm{Log}\,\|S_n x\| = \gamma_1 \qquad , \quad \text{a.s.} \quad .$$

The other results require a detailed study of the Markov chain $(Y_{i+1}, \overline{S_i x})$. Its essential properties will be derived from the fact that, loosely speaking, the random matrices S_n have asymptocally a contracting action on the directions. To be more precise, let δ be the natural angular distance on $P(\mathbb{R}^d)$

$$\delta(\overline{x},\overline{y}) = |\sin\{\mathrm{angle}(\overline{x},\overline{y})\}| \ .$$

We shall prove, following Guivarc'h and Raugi [34], that under fairly weak assumptions,

$$\limsup_{n\to\infty} \frac{1}{n}\,\mathrm{Log}\,\delta(\overline{S_n x}, \overline{S_n y}) < 0 \qquad \text{a.s.} \qquad (1)$$

for any fixed $\overline{x}$ and $\overline{y}$ in $P(\mathbb{R}^d)$.

This property is of primary importance. For instance it yields easily that the Markov chain $(Y_{i+1}, \overline{S_i x})$ has a unique invariant probability measure. More crucially we shall see, following Le Page [49] , that (1) is the cornerstone on which rest the proofs of the central limit theorem, the estimate for large deviations and many further results. We might say that, in some sense, this inequality replaces Doeblin's condition under which the precise limit theorems for functionalsof Markov chains are usually available.

Let $\gamma_1 \geqq \gamma_2 \geqq \cdots \geqq \gamma_d$ be the Lyapunov exponents associated with the sequence (Y_n) . Since

$$\limsup_{n\to\infty} \frac{1}{n} \operatorname{Log} \delta(\overline{S_n x}, \overline{S_n y}) \leqq \gamma_2 - \gamma_1$$

our first task is to show that $\gamma_1 > \gamma_2$.

The main purpose of the first three chapters is to derive the inequality $\gamma_1 > \gamma_2$ from a careful analysis of the qualitative behaviour of the Markov chain $(\overline{S_n x})$.

In Chapter I we introduce the basic properties of the upper Lyapunov exponent γ_1 .

In Chapter II we restrict ourselves to 2×2 matrices. For pedagogical purpose we first develop in this simple setting the general argument leading to a proof of this inequality. Most of the results which are needed in part B can already be found here.

Chapter III treats the case of matrices of arbitrary order.

In Chapter IV we digress from the main line. We apply the preceding results to the study of all the Lyapunov exponents. We also briefly present the link between these exponents and the boundary theory of Furstenberg.

In Chapter V we derive the main limit theorem on $\operatorname{Log} \|S_n x\|$ from the inequality $\gamma_1 > \gamma_2$. Further results, such as limit theorems for the coefficients of the matrices S_n , are proved in Chapter VI.

All the main results proved in this text come from Furstenberg [21] , Guivarc'h and Raugi [34] and Le Page [49] . We hope that our exposition entices the reader to go back to these profound original works.

Despite its importance, we have chosen not to consider Osseledec's theorem. The reason is twofold. Firstly we have tried to keep the prerequisites to a minimum and to give a self-contained account of the subject. Secondly Ledrappier has already given in [46] a beautiful treatment of the applications of this theorem to products of i.i.d. random matrices. We felt there was nothing to gain by a reproduction of this material. In our opinion, the reader who wants to have a full picture of random products has to read Ledrappier's monograph. It will then be an easy and useful exercise to him to check how some of our proofs can be shortened by making use of his results.

We have not considered positive matrices (except in some exercises). We understand that Joel Cohen is writing a book on this subject.

For the sake of simplicity of notation we have restricted ourselves to matrices with real entries. But all the statements are also true in the complex case (the proofs carry over immediately to this case by replacing everywhere $\mathbb{R}$ by $\mathbb{C}$) .

Some chapters contain "complements" sections. They develop some additional material which is not used elsewhere and the reader can skip them. Some of their proofs are only outlined and may be of a more advanced level.