

Applications of  
Mathematics  
*Stochastic Modelling  
and Applied Probability*

24

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Harold J. Kushner     Paul G. Dupuis

# Numerical Methods for Stochastic Control Problems in Continuous Time

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Harold J. Kushner  
Division of Applied Mathematics  
Brown University  
Providence, RI 02912 USA

Paul G. Dupuis  
Division of Applied Mathematics  
Brown University  
Providence, RI 02912 USA

*Managing Editors*

A.V. Balakrishnan  
Electrical Engineering Department  
University of California  
Los Angeles, CA 90024 USA

I. Karatzas  
Department of Statistics  
Columbia University  
New York, NY 10027 USA

M. Yor  
Laboratoire de Probabilités  
Université Pierre et Marie Curie  
4 Place Jussieu, Tour 56  
75230 Paris Cedex  
France

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*To*  
*Linda, Diana, and Nina*  
*and*  
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