Applied Probability
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15

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Stochastic Storage Processes

Queues, Insurance Risk, and Dams

Springer-Verlag
New York Heidelberg Berlin

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AMS Subject Classifications (1980): Primary 60K30; Secondary 60G20, 60K25, 62P05, 62P99

Library of Congress Cataloging in Publication Data

Prabhu, Narahari Umanath, 1924-
Stochastic storage processes.

(Applications of mathematics; 15)
Bibliography: p.
Includes index.
1. Queuing theory. 2. Stochastic processes.
I. Title.
T57.9.P63 519.8′2 80-18686

Softcover reprint of the hardcover 1st edition 1980

ISBN-13: 978-1-4684-0115-8 e-ISBN-13: 978-1-4684-0113-4
DOI: 10.1007/978-1-4684-0113-4

To All of My Students

Preface

This book is based on a course I have taught at Cornell University since 1965. The primary topic of this course was queueing theory, but related topics such as inventories, insurance risk, and dams were also included. As a text I used my earlier book, *Queues and Inventories* (John Wiley, New York, 1965). Over the years the emphasis in this course shifted from detailed analysis of probability models to the study of stochastic processes that arise from them, and the subtitle of the text, "A Study of Their Basic Stochastic Processes," became a more appropriate description of the course. My own research into the fluctuation theory for Lévy processes provided a new perspective on the topics discussed, and enabled me to reorganize the material. The lecture notes used for the course went through several versions, and the final version became this book. A detailed description of my approach will be found in the Introduction.

I have not attempted to give credit to authors of individual results. Readers interested in the historical literature should consult the Selected Bibliography given at the end of the Introduction. The original work in this area is presented here with simpler proofs that make full use of the special features of the underlying stochastic processes. The same approach makes it possible to provide several new results.

Thanks are due to Kathy King for her excellent typing of the manuscript. To my wife Sumi goes my sincere appreciation for her constant encouragement, especially during the preparation of the last chapter. Finally, I have dedicated this book to all of my students: their questions, comments, and criticism have made a significant contribution to the presentation.

Ithaca, New York
November 1980

N. U. Prabhu

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Abbreviations and Notation

Abbreviations

Term	*Abbreviation*
Characteristic function	c.f.
Distribution function	d.f.
If and only if	Iff
Laplace transform	L.T.
Probability generating function	p.g.f.

The term *transform* is used for expressions such as $E(z^N e^{-\theta X})$ where N is integer-valued and $X \geq 0$. The notation F is used both for a distribution measure and a d.f., leading to the use of $F(dx)$ and $dF(x)$ respectively.

Notations

1. *The normal d.f.*

$$N(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-1/2y^2}\, dy(-\infty < x < \infty).$$

2. *One sided normal d.f.'s*

(i)

$$N_+(x) = 0 \qquad \text{for } x \leq 0,$$
$$N_+(x) = 2N(x) - 1 \qquad \text{for } x \geq 0.$$

This distribution has mean $= \sqrt{2/\pi}$ and variance $= 1 - 2/\pi$.

(ii)
$$N_-(x) = 2N(x) \qquad \text{for } x \le 0$$
$$N_-(x) = 1 \qquad \text{for } x \ge 0.$$

If the random variable x has d.f. N_+, then $-x$ has d.f. N_-.

3. *Stable d.f. with exponent* $\frac{1}{2}$

$$G_{1/2}(x) = 0 \qquad \text{for } x \le 0,$$
$$G_{1/2}(x) = 2\left[1 - N\left(\frac{1}{\sqrt{x}}\right)\right] \qquad \text{for } x \ge 0.$$

This distribution is more easily recognized by its density, which is given by

$$g_{1/2}(x) = 0 \qquad \text{for } x \le 0,$$
$$g_{1/2}(x) = \frac{1}{\sqrt{2\pi x^3}} e^{-1/2x} \qquad \text{for } x \ge 0$$

or by its Laplace transform, which is $e^{-\sqrt{2\theta}}$ $(\theta > 0)$. The mean of the distribution is ∞.