Topological and Variational Methods with Applications to Nonlinear Boundary Value Problems

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Preface

This monograph presents fundamental methods and topics in nonlinear analysis and their efficient application to nonlinear boundary value problems for elliptic equations. The book is divided into 12 chapters, with 9 chapters covering the theoretical material – Sobolev spaces, nonlinear operators, nonsmooth analysis, degree theory, variational principles and critical point theory, Morse theory, bifurcation theory, regularity theorems and maximum principles, and spectrum of differential operators – followed by three chapters containing applications to ordinary differential equations and nonlinear elliptic equations with Dirichlet or Neumann boundary conditions. The last three chapters, but not only those, consist to a large extent of original results due to the authors, and many of these results appear here in a novel form, with significant improvements and developments. We emphasize that the first nine chapters devoted to general theories are not just a collection of relevant tools to study the nonlinear boundary value problems considered in the last three chapters. They offer broad and essential insight into powerful abstract theories. Major objectives for us have been to make a self-contained presentation for every treated subject and show that it applies to different types of problems.

This book originated in the collaboration of the three authors that gave rise during a period of about 10 years to a series of research papers studying nonlinear boundary value problems with Dirichlet and Neumann boundary conditions and having in the differential part Laplacian, p-Laplacian, or, more generally, even nonhomogeneous differential operators. These papers are reflected in our book, although the initial results are mostly rewritten, revised, and sharpened in the text here. A distinct feature of our work is that we combine various methods such as nonlinear operator theories, degree theory, lower and upper solutions, variational methods, Morse theory, regularity, maximum principles, and spectral theory. For instance, this can be seen in the study of multiple solutions, where every solution is usually obtained through a different approach and method.

The material in our book mainly addresses researchers in pure and applied mathematics, physics, mechanics, and engineering. It is also accessible to graduate students in mathematical and applied sciences, who will find updated information and a systematic exposition of important parts of modern mathematics.

The authors are deeply indebted to Dr. Lucas Fresse for his immense and generous help related to the whole work for the present book. We have decisively benefited from his brilliant ideas and insight. For instance, the version of the first deformation theorem and its application to the limit case in the minimax principle, Lemma 6.65, which is helpful in our presentation of Morse theory, the general version of the Moser iteration procedure, and the unified formulation of the local minimizer principle given in Theorem 12.18 are due to him. His outstanding contributions in improving every chapter of our book are gratefully acknowledged.

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Introduction

Nonlinear elliptic boundary value problems have proven to be an extremely fruitful area of application of topological and variational methods. Such methods usually exploit the special form of the nonlinearities entering the problem, for instance their symmetries, and offer complementary information. They are powerful tools to show the existence of multiple solutions and establish qualitative results on these solutions, for instance information regarding their location. The topological and variational approach provides not just the existence of a solution, usually several solutions, but allows one to acquire relevant knowledge about the behavior and properties of the solutions, which is extremely useful because generally the problems cannot be effectively solved, so the precise expression of the solutions is unknown. As a specific example of a property of a solution that we look for is the sign of the solution, for example, to be able to determine whether it is positive, negative, or nodal (i.e., sign changing). Such topics will be addressed in the present work.

The aim of this monograph is twofold: (1) to present, in a rigorous, modern, and coherent way, topological and variational methods from the point of view of nonlinear analysis; (2) to study nonlinear elliptic boundary value problems in order to infer qualitative properties of the solutions. These two major goals strongly interact. On the one hand, topological and variational methods enable us to discover important information about solutions, including their existence. On the other hand, investigation of the nonlinear elliptic boundary value problems illustrates and justifies in an ideal manner the power of the abstract techniques developed through topological and variational methods. Our book is based on the idea of capturing this close relationship and is designed to maintain a unifying treatment. Actually, the study of every topic considered here relies on previous results that can be found in the body of the book. In this sense, our work is self-contained. To show the unity of the book, we mention a few aspects that will be encountered in our text. The topological degree is used to investigate the linking properties related to critical point theory, also in bifurcation theory, in handling different nonlinear elliptic equations. Nonsmooth analysis is utilized to study boundary value problems with multivalued terms. Minimax results, such as the celebrated mountain pass and saddle point theorems, are extremely useful in handling nonlinear boundary value problems exhibiting a variational structure. The bifurcation theory permits us to deal with nonlinear equations depending on parameters. Various index theories, such as the genus, lead to multiplicity results for the solutions. Minimization with constraints, for instance on submanifolds, provides spectral information about nonlinear operators and is exploited for treating nonlinear boundary value problems subject to additional restrictions. There are many other situations that show the unifying character of our work.

The book consists of nine chapters devoted to abstract topological and variational methods and three chapters focusing on boundary value problems for nonlinear elliptic equations. It is worth mentioning that the first nine chapters are projected not just to provide the mathematical background to be applied in the last three chapters. They are of independent interest and give a comprehensive account, sometimes with traits of originality, of large areas of contemporary mathematics. Moreover, in many applications considered in our work, we make use of several methods for the same problem: minimization, variational principle, minimax methods, degree theory, lower and upper solutions, nonlinear operators, Morse theory, regularization, truncation, and maximum principle. Except in the first chapter, we provide complete proofs for almost all of the stated results, in many situations simplifying the arguments, sometimes correcting obscurities that exist in different references, or from being able to have hypotheses that are more general than those known from other works and even to improve the conclusions. Many examples are given in the text to illustrate the applicability of the abstract statements, especially in the parts focusing on applications to nonlinear boundary value problems. Every chapter has a final section where we indicate the relevance of the discussed topics, related references, and our specific contributions with respect to the available literature.

Finally, we briefly describe the contents of the chapters in the book. A more detailed description can be found in the abstracts appearing at the beginning of the chapters. Chapter 1 reports on the background of Sobolev spaces that is needed in the sequel. Chapter 2 discusses important classes of nonlinear operators: compact, maximal monotone, pseudomonotone, generalized pseudomonotone, $(S)_+$-maps, and Nemytskii operators. Chapter 3 has as its object convex analysis and subdifferentiability theory for locally Lipschitz functions. Chapter 4 presents degree theories: Brouwer's degree, Leray–Schauder degree, degree for $(S)_+$-maps, and degree for set-valued maps. Chapter 5 addresses variational principles and critical point theory, including minimax theorems formulated both for smooth and nonsmooth functions. Chapter 6 sets forth the basic facts of Morse theory emphasizing the study of critical groups. Chapter 7 highlights bifurcation results for parametric equations obtained through degree theory and the implicit function theorem. Chapter 8 consists of basic results in regularity theory and maximum principles for nonlinear elliptic equations. Chapter 9 is devoted to the spectral properties of some fundamental differential operators: Laplacian, p-Laplacian, and p-Laplacian plus an indefinite potential. Chapter 10 focuses on the periodic solutions of nonlinear ordinary differential equations. Chapter 11 examines nonlinear Dirichlet boundary value problems in a multitude of cases, such as sublinear, asymptotically linear, superlinear, coercive, noncoercive, parametric, resonant, and near resonant, and through various methods such as degree theory, variational methods, lower and upper solutions, Morse theory, and nonlinear operators techniques. Chapter 12 contains recent results on nonlinear elliptic equations with Neumann boundary conditions, pointing out advances in topics such as resonance from the left and from the right for Neumann problems depending on a parameter, Neumann equations whose differential part is expressed by means of a nonhomogeneous operator, and a unifying approach to sublinear and superlinear cases for semilinear Neumann problems. A list of symbols, references, and an index conclude the book.