Part IV
Paradoxes and Axioms

Introduction to Part IV

In Parts I–III of this book we developed cardinals, order, ordinals, and real point set theory, and also indicated how these relate to some classical areas of mathematics. We carried out that development in an informal and naive way as we would do for any standard area of mathematics such as geometry, exploring structural details and obtaining views and intuitions about the subject matter. In this sense, Parts I–III of the book were purely mathematical.

The naive theory of sets, however, can lead to contradictions unless suitable restrictions are placed on the simple principles forming the basis of the theory. This requires a careful scrutiny of the logical foundations of set theory. To stay focused on the mathematical aspects of our topics, we had so far avoided getting into this *metamathematical* problem.

In this part, we will give an overview of such logical and foundational matters, starting with some famous contradictions of naive set theory and two early responses to them (Chapter 20). Our coverage will necessarily be very elementary and introductory, and we will refer the reader to more comprehensive works for further details.

In Chap. 21, we briefly present Zermelo–Fraenkel set theory (ZF) and the von Neumann ordinals, providing only bare outlines for the formal development of some of the basic notions of set theory such as order and cardinals. However, the reader who has mastered the theories of numbers, cardinals, ordinals, and the real continuum developed in Parts I–III, will find the re-development of all these theories within the formal framework of ZF a relatively routine matter, and we encourage the reader to take up this project of replicating the results of Parts I–III formally in ZF.

Finally, the postscript to this part (Chapter 22) provides glimpses into some landmark results of set theory of the past 75 years.