Part III
Real Point Sets

Introduction to Part III

This part focuses exclusively on the real line **R**. Cantor's work not only gave birth to the theory of transfinite, but was also instrumental in the development of *point set topology,* which, roughly speaking, is the study of limits and continuity in a general setting. Topological notions such as closed sets, dense-in-itself sets, and perfect sets were first introduced by Cantor.

The opening chapter, much of which is very elementary, introduces base representation via interval trees, *Cantor systems,* and *generalized Cantor sets.* The next chapter deals with basic topology of the real line.

The material of the chapter on Heine–Borel and Baire-Category Theorems is often called "measure and category." It is shown that G_δ sets satisfy the Continuum Hypothesis, and that perfect sets have cardinality $\mathfrak{c}$.

The chapters on Cantor–Bendixson analysis and on Brouwer's and Sierpinski's Theorems are somewhat more special. An application of the ordinals is illustrated by the method of Cantor–Bendixson analysis, giving a complete enumeration of the $\aleph_1$ distinct "homeomorphism types" of countable compact sets. The proofs of Brouwer's and Sierpinski's Theorems given here illustrate how the Cantor–Dedekind theory of order can be used to give somewhat elementary proofs of some relatively advanced topological results.

The chapter on Borel and analytic sets touches on the rudiments of descriptive set theory, and proves that the analytic sets have the perfect set property—the best possible result that can be proved using the usual axioms of set theory. They are also shown to be Lebesgue measurable (and having the Baire property) using the Ulam matrix decomposition for coanalytic sets. To obtain a non-Borel analytic set, a direct effective proof of the boundedness theorem for the set of codes of well-founded trees is given (since with no access to product spaces, the standard method of diagonalizing universal sets cannot be used).

The postscript chapter for this part gives a detailed account of Ulam's analysis of the measure problem leading to the notion of measurable cardinals, and a brief discussion of Lusin's problem for the projective sets.