

# Mathematics for Econometrics



Phoebus J. Dhrymes

# Mathematics for Econometrics

Fourth Edition



Springer

Phoebus J. Dhrymes  
Department of Economics  
Columbia University  
New York, USA

ISBN 978-1-4614-8144-7      ISBN 978-1-4614-8145-4 (eBook)  
DOI 10.1007/978-1-4614-8145-4  
Springer New York Heidelberg Dordrecht London

Library of Congress Control Number: 2013944568

© The Author 2013

This work is subject to copyright. All rights are reserved by the Publisher, whether the whole or part of the material is concerned, specifically the rights of translation, reprinting, reuse of illustrations, recitation, broadcasting, reproduction on microfilms or in any other physical way, and transmission or information storage and retrieval, electronic adaptation, computer software, or by similar or dissimilar methodology now known or hereafter developed. Exempted from this legal reservation are brief excerpts in connection with reviews or scholarly analysis or material supplied specifically for the purpose of being entered and executed on a computer system, for exclusive use by the purchaser of the work. Duplication of this publication or parts thereof is permitted only under the provisions of the Copyright Law of the Publisher's location, in its current version, and permission for use must always be obtained from Springer. Permissions for use may be obtained through RightsLink at the Copyright Clearance Center. Violations are liable to prosecution under the respective Copyright Law.

The use of general descriptive names, registered names, trademarks, service marks, etc. in this publication does not imply, even in the absence of a specific statement, that such names are exempt from the relevant protective laws and regulations and therefore free for general use.

While the advice and information in this book are believed to be true and accurate at the date of publication, neither the authors nor the editors nor the publisher can accept any legal responsibility for any errors or omissions that may be made. The publisher makes no warranty, express or implied, with respect to the material contained herein.

Printed on acid-free paper

Springer is part of Springer Science+Business Media ([www.springer.com](http://www.springer.com))

# Preface

The fourth edition differs significantly from the third edition, in that it has undergone considerable expansion and revision.

The major expansion involves a more complete coverage of basic aspects of mathematics that have continued to play an increasingly significant role in the literature of econometrics. Thus, the chapter on difference equations has been expanded to include enhanced treatment of lag operators (backward shift operators in the statistical literature) that are important not only in the context of the dynamic simultaneous equation GLSEM (general linear structural econometric model), but also time series analysis.

In addition, a chapter on the basic mathematics underlying the analytics of probability theory has been added, as well as a chapter on laws of large numbers and central limit theorems that form the probabilistic basis of classical econometrics. Moreover, there is an informative but not exhaustive discussion of stationary time series analysis, including discussions of the taxonomy of time series, issues of causality and invertibility, with a limited treatment of certain non-linearities such as those found in the popular ARCH (autoregressive conditional heteroskedasticity) model, which together with its many variants has found extensive applications in the literature of financial econometrics. However, there is no discussion of non-stationary time series, which is the subject of the author's *Time Series Unit Roots and Cointegration*, Academic Press, 1998.

Finally, this edition contains two fairly extensive chapters on applications to the GLM (general linear model), GLSEM and time series analysis which treat issues relevant to their underlying theoretical bases, estimation and forecasting.

New York, USA

Phoebus J. Dhrymes



# Preface to the Third Edition

The third edition differs from the second edition in several respects. The coverage of matrix algebra has been expanded. For example, the topic of inverting partitioned matrices in this edition deals explicitly with a problem that arises in estimation under (linear) constraints. Often this problem forces us to deal with a block partitioned matrix whose (1,1) and (2,2) blocks are singular matrices. The standard method for inverting such matrices fails; unless the problem is resolved, explicit representation of estimators and associated Lagrange multipliers is not available. An important application is in estimating the parameters of the general linear structural econometric model, when the identifying restrictions are imposed by means of Lagrange multipliers. This formulation permits a near effortless test of the validity of such (overidentifying) restrictions.

This edition also contains a treatment of the vector representation of restricted matrices such as symmetric, triangular, diagonal and the like. The representation is in terms of restricted linear subspaces. Another new feature is the treatment of permutation matrices and the  $\text{vec}$  operator, leading to an explicit representation of the relationship between  $A \otimes B$  and  $B \otimes A$ .

In addition, it contains three new chapters, one on asymptotic expansions and two on applications of the material covered in this volume to the general linear model and the general linear structural econometric model, respectively. The salient features of the estimation problems in these two topics are discussed rigorously and succinctly.

This version should be useful to students and professionals alike as a ready reference to mathematical tools and results of general applicability

in econometrics. The two applications chapters should also prove useful to noneconomist professionals who are interested in gaining some understanding of certain topics in econometrics.

New York, USA

Phoebus J. Dhrymes



# Preface to the Second Edition

The reception of this booklet has encouraged me to prepare a second edition.

The present version is essentially the original, but adds a number of very useful results in terms of inverses and other features of partitioned matrices, a discussion of the singular value decomposition for rectangular matrices, issues of stability for the general linear structural econometric model, and similar topics.

I would like to take this opportunity to express my thanks to many of my students and others for pointing out misprints and incongruities in the first edition.

New York, USA

Phoebus J. Dhrymes



# Preface to the First Edition

This book began as an Appendix to *Introductory Econometrics*. As it progressed, requirements of consistency and completeness of coverage seemed to make it inordinately long to serve merely as an Appendix, and thus it appears as a work in its own right.

Its purpose is not to give rigorous instruction in mathematics. Rather it aims at filling the gaps in the typical student's or professional's mathematical training, to the extent relevant for the study of econometrics.

Thus, it contains a collection of mathematical results employed at various stage of *Introductory Econometrics*. More generally, however, it could serve as a useful adjunct and reference to students of econometrics, no matter what text is being employed.

In the vast majority of cases, proofs are provided and there is a modicum of verbal discussion of certain mathematical results, the objective being to reinforce the student's understanding of the formalities. In certain instances, however, when proofs are too cumbersome, or complex, or when they are too obvious, they are omitted.

New York, USA

Phoebus J. Dhrymes



# Contents

<b>Preface</b>	<b>v</b>
<b>Preface to the Third Edition</b>	<b>vii</b>
<b>Preface to the Second Edition</b>	<b>ix</b>
<b>Preface to the First Edition</b>	<b>xi</b>
<b>1 Vectors and Vector Spaces</b>	<b>1</b>
1.1 Complex Numbers and Vectors . . . . .	1
1.1.1 Polar Form of Complex Numbers . . . . .	3
1.2 Vectors . . . . .	5
1.3 Vector Spaces . . . . .	7
1.3.1 Basis of a Vector Space . . . . .	8
1.4 Subspaces of a Vector Space . . . . .	9
<b>2 Matrix Algebra</b>	<b>13</b>
2.1 Basic Definitions . . . . .	13
2.2 Basic Operations . . . . .	15
2.3 Rank and Inverse of a Matrix . . . . .	17
2.3.1 Matrix Inversion . . . . .	18
2.4 Hermite Forms and Rank Factorization . . . . .	23
2.4.1 Rank Factorization . . . . .	28
2.5 Trace and Determinants . . . . .	30
2.6 Computation of the Inverse . . . . .	39
2.7 Partitioned Matrices . . . . .	40
2.8 Kronecker Products of Matrices . . . . .	51

2.9	Characteristic Roots and Vectors . . . . .	54
2.9.1	Kronecker Product Matrices . . . . .	67
2.10	Orthogonal Matrices . . . . .	68
2.11	Symmetric Matrices . . . . .	71
2.12	Idempotent Matrices . . . . .	79
2.13	Semi-definite and Definite Matrices . . . . .	80
<b>3</b>	<b>Systems of Linear Equations</b>	<b>95</b>
3.1	Introduction . . . . .	95
3.2	The c-, s-, and g-Inverses . . . . .	96
3.3	Properties of the Generalized Inverse . . . . .	100
3.4	Linear Equations and Pseudoinverses . . . . .	108
3.5	Approximate Solutions . . . . .	112
<b>4</b>	<b>Matrix Vectorization</b>	<b>119</b>
4.1	Introduction . . . . .	119
4.2	Vectorization of Matrices . . . . .	120
4.3	Linearly Restricted Matrices . . . . .	124
4.3.1	Restricted Subspaces . . . . .	128
4.3.2	The Structure of Restricted Subspaces . . . . .	137
4.3.3	Permutation Matrices and the vec Operator . . . . .	139
4.3.4	Permutation and the vec Operator . . . . .	147
<b>5</b>	<b>Vector and Matrix Differentiation</b>	<b>149</b>
5.1	Introduction . . . . .	149
5.2	Derivatives of Functions of the Form $y = Ax$ . . . . .	150
5.3	Derivatives of Functions of the Form $y = z'Ax$ . . . . .	153
5.4	Differentiation of the Trace . . . . .	157
5.5	Differentiation of Determinants . . . . .	161
5.6	Differentiation of Inverse of a Matrix . . . . .	169
<b>6</b>	<b>DE Lag Operators GLSEM and Time Series</b>	<b>171</b>
6.1	The Scalar Second-Order Equation . . . . .	171
6.2	The Lag Operator L and Its Algebra . . . . .	173
6.3	Vector Difference Equations . . . . .	176
6.3.1	Factorization of High Order Polynomials . . . . .	177
6.4	Applications . . . . .	180
6.4.1	Preliminaries . . . . .	180
6.4.2	GLSEM . . . . .	181
6.4.3	Applications to Time Series Models . . . . .	185
6.4.4	An $AR(m)$ with Nonlinearities; ARCH Models . . . . .	192

<b>7</b>	<b>Mathematical Underpinnings of Probability Theory</b>	<b>197</b>
7.1	Sets and Set Operations . . . . .	197
7.2	Limits of Sequences . . . . .	199
7.3	Measurable Spaces, Algebras and Sets . . . . .	204
7.4	Measures and Probability Measures . . . . .	209
7.4.1	Measures and Measurable Functions . . . . .	209
7.5	Integration . . . . .	216
7.5.1	Miscellaneous Convergence Results . . . . .	221
7.6	Extensions to Abstract Spaces . . . . .	227
<b>8</b>	<b>Foundations of Probability</b>	<b>235</b>
8.1	Discrete Models . . . . .	235
8.2	General Probability Models . . . . .	238
8.2.1	The Measurable Space $(R^n, \mathcal{B}(R^n))$ . . . . .	238
8.2.2	Specification of Probability Measures . . . . .	242
8.2.3	Fubini's Theorem and Miscellaneous Results . . . . .	251
8.3	Random Variables . . . . .	255
8.3.1	Generalities . . . . .	255
8.3.2	Moments of Random Variables and Miscellaneous Inequalities . . . . .	258
8.4	Conditional Probability . . . . .	263
8.4.1	Conditional Probability in Discrete Models . . . . .	263
8.4.2	Conditional Probability in Continuous Models . . . . .	273
8.4.3	Independence . . . . .	277
<b>9</b>	<b>LLN, CLT and Ergodicity</b>	<b>285</b>
9.1	Review and Miscellaneous Results . . . . .	285
9.1.1	Limits of sets . . . . .	286
9.1.2	Modes of Convergence . . . . .	288
9.1.3	Convergence in Probability . . . . .	288
9.1.4	Convergence in Mean of Order $p$ , or $L^p$ . . . . .	288
9.1.5	Convergence in Distribution . . . . .	289
9.2	Relationship Among Modes of Convergence . . . . .	290
9.2.1	Applications of Modes of Convergence . . . . .	290
9.3	Laws of Large Numbers and Central Limit Theorems . . . . .	292
9.3.1	Criteria for Applicability, LLN . . . . .	293
9.3.2	Criteria for Applicability, CLT . . . . .	294
9.4	Ergodicity and Applications . . . . .	298
9.4.1	Preliminaries . . . . .	298
9.4.2	Ergodicity . . . . .	300
9.4.3	Applications . . . . .	304

<b>10 The General Linear Model</b>	<b>309</b>
10.1 Introduction . . . . .	309
10.2 The GLM and the Least Squares Estimator . . . . .	310
10.3 Properties of the Estimators . . . . .	312
10.4 Distribution of OLS Estimators . . . . .	314
10.4.1 A Digression on the Multivariate Normal . . . . .	314
10.4.2 Application to OLS Estimators . . . . .	316
10.5 Nonstandard Errors . . . . .	321
10.6 Inference with Asymptotics . . . . .	325
10.7 Orthogonal Regressors . . . . .	328
10.8 Multiple GLM . . . . .	332
<b>11 Panel Data Models</b>	<b>335</b>
11.1 Introduction . . . . .	335
11.2 Notation . . . . .	336
11.3 Interpretation and Assumptions . . . . .	338
11.3.1 Interpretation . . . . .	338
11.3.2 Assumptions . . . . .	339
11.4 Estimation . . . . .	341
11.4.1 Within Groups Estimation . . . . .	341
<b>12 GLSEM and TS Models</b>	<b>351</b>
12.1 GLSEM . . . . .	351
12.1.1 Preliminaries . . . . .	351
12.1.2 The Structural and Reduced Forms . . . . .	351
12.1.3 Inconsistency of Ordinary Least Squares . . . . .	354
12.2 Consistent Estimation: 2SLS and 3SLS . . . . .	356
12.2.1 An Alternative Derivation of 2SLS and 3SLS . . . . .	357
12.2.2 Systemwide Estimation . . . . .	358
12.3 Identification Issues . . . . .	362
12.3.1 Tests of Over-identifying Restrictions . . . . .	364
12.3.2 An Alternative Approach to Identification . . . . .	370
12.4 The Structural VAR Model . . . . .	374
12.5 Forecasting from GLSEM and TS Models . . . . .	378
12.5.1 Forecasting with the GLSEM . . . . .	378
12.5.2 Forecasting with Time Series Models (TSM) . . . . .	384
<b>13 Asymptotic Expansions</b>	<b>393</b>
13.1 Introduction . . . . .	393
13.2 Preliminaries . . . . .	394
13.3 General Bounds on Approximations . . . . .	396



13.4 Characteristic Functions (CF) and Moment Generating Functions (MGF) . . . . .	397
13.5 CF, MGF, Moments, and Cumulants . . . . .	401
13.5.1 CF, MGF, and the Existence of Moments . . . . .	401
13.5.2 Cumulants and Cumulant Generating Functions (CGF) . . . . .	406
13.6 Series Approximation . . . . .	408
<b>Bibliography</b>	<b>411</b>
<b>Index</b>	<b>413</b>