Graduate Texts in Mathematics 225

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Lie Groups

Second Edition

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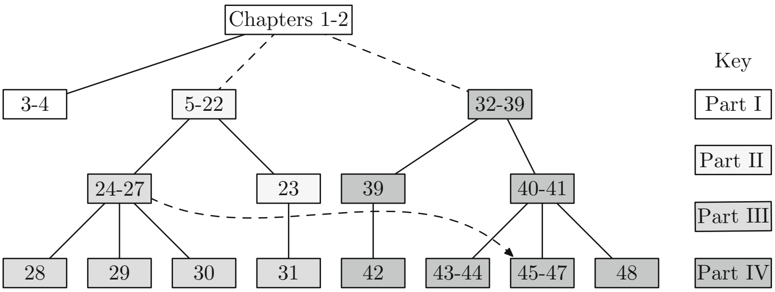
Preface

This book aims to be both a graduate text and a study resource for Lie groups. It tries to strike a compromise between accessibility and getting enough depth to communicate important insights. In discussing the literature, often secondary sources are preferred: cited works are usually recommended ones.

There are four parts. Parts I, II or IV are all "starting points" where one could begin reading or lecturing. On the other hand, Part III assumes familiarity with Part II. The following chart indicates the approximate dependencies of the chapters. There are other dependencies, where a result is used from a chapter that is not a prerequisite according to this chart: but these are relatively minor. The dashed lines from Chaps. 1 and 2 to the opening chapters of Parts II and IV indicate that the reader will benefit from knowledge of Schur orthogonality but may skip or postpone Chaps. 1 and 2 before starting Part II or Part IV. The other dashed line indicates that the Bruhat decomposition (Chap. 27) is assumed in the last few chapters of Part IV.



The two lines of development in Parts II–IV were kept independent because it was possible to do so. This has the obvious advantage that one may start reading with Part IV for an alternative course. This should not obscure the fact that these two lines are complementary, and shed light on each other. We hope the reader will study the whole book.

Part I treats two basic topics in the analysis of compact Lie groups: Schur orthogonality and the Peter–Weyl theorem, which says that the irreducible unitary representations of a compact group are all finite-dimensional.

Usually the study of Lie groups begins with compact Lie groups. It is attractive to make this the complete content of a short course because it can be treated as a self-contained subject with well-defined goals, about the right size for a 10-week class. Indeed, Part II, which covers this theory, could be used as a traditional course culminating in the Weyl character formula. It covers the basic facts about compact Lie groups: the fundamental group, conjugacy of maximal tori, roots and weights, the Weyl group, the Weyl integration formula, and the Weyl character formula. These are basic tools, and a short course in Lie theory might end up with the Weyl character formula, though usually I try to do a bit more in a 10-week course, even at the expense of skipping a few proofs in the lectures. The last chapter in Part II introduces the affine Weyl group and computes the fundamental group. It can be skipped since Part III does not depend on it.

`Sage`, the free mathematical software system, is capable of doing typical Lie theory calculations. The student of Part II may want to learn to use it. An appendix illustrates its use.

The goal of Part I is the Peter–Weyl theorem, but Part II does not depend on this. Therefore one could skip Part I and start with Part II. Usually when I teach this material, I do spend one or two lectures on Part I, proving Schur orthogonality but not the Peter–Weyl formula. In the interests of speed I tend to skip a few proofs in the lectures. For example, the conjugacy of maximal tori needs to be proved, and this depends in turn on the surjectivity of the exponential map for compact groups, that is, Theorem 16.3. This is proved completely in the text, and I think it should be proved in class but some of the differential geometry details behind it can be replaced by intuitive explanations. So in lecturing, I try to explain the intuitive content of the proof without going back and proving Proposition 16.1 in class. Beginning with Theorems 16.2–16.4, the results to the end of the chapter, culminating in various important facts such as the conjugacy of maximal tori and the connectedness of centralizers can all be done in class. In the lectures I prove the Weyl integration formula and (if there is time) the local Frobenius theorem. But I skip a few things like Theorem 13.3. Then it is possible to get to the Weyl Character formula in under 10 weeks.

Although compact Lie groups are an essential topic that can be treated in one quarter, noncompact Lie groups are equally important. A key role in much of mathematics is played by the Borel subgroup of a Lie group. For example, if $G = \mathrm{GL}(n, \mathbb{R})$ or $\mathrm{GL}(n, \mathbb{C})$, the Borel subgroup is the subgroup of upper triangular matrices, or any conjugate of this subgroup. It is involved in two important results, the Bruhat and Iwasawa decompositions. A noncompact Lie group has two important classes of homogeneous spaces, namely symmetric spaces and flag varieties, which are at the heart of a great deal of important modern mathematics. Therefore, noncompact Lie groups cannot be ignored, and we tried hard to include them.

In Part III we first introduce a class of noncompact groups, the complex reductive groups, that are obtained from compact Lie groups by "complexification." These are studied in several chapters before eventually taking on general noncompact Lie groups. This allows us to introduce key topics such as the Iwasawa and Bruhat decompositions without getting too caught up in technicalities. Then we look at the Weyl group and affine Weyl group, already introduced in Part II, as Coxeter groups. There are two important facts about them to be proved: that they have Coxeter group presentations, and the theorem of Matsumoto and Tits that any two reduced words for the same element may be related by applications of the braid relations.

For these two facts we give geometric proofs, based on properties of the complexes on which they act. These complexes are the system of Weyl chambers in the first case, and of alcoves in the second. Applications are given, such as Demazure characters and the Bruhat order. For complex reductive groups, we prove the Iwasawa and Bruhat decompositions, digressing to discuss some of the implications of the Bruhat decomposition for the flag manifold. In particular the Schubert and Bott–Samelson varieties, the Borel-Weil theorem and the Bruhat order are introduced. Then we look at symmetric spaces, in a chapter that alternates examples with theory. Symmetric spaces occur in pairs, a compact space matched with a noncompact one. We see how some symmetric spaces, the Hermitian ones, have complex structures and are important in the theory of functions of several complex variables. Others are convex cones. We take a look at Freudenthal's "magic square." We discuss the embedding of a noncompact symmetric space in its compact dual, the boundary components and Bergman–Shilov boundary of a symmetric tube domain, and Cartan's classification. By now we are dealing with arbitrary noncompact Lie groups, where before we limited ourselves to the complex analytic ones. Another chapter constructs the relative root system, explains Satake diagrams and gives examples illustrating the various phenomena that can occur. The Iwasawa decomposition, formerly obtained for complex analytic groups, is reproved in this more general context. Another chapter surveys the different ways Lie groups can be embedded in one another. Part III ends with a somewhat lengthy discussion of the spin representations of the double covers of orthogonal groups. First, we consider what can be deduced from the Weyl theory. Second, as an alternative, we construct the spin representations using Clifford algebras. Instead of following the approach (due to Chevalley) often taken in embedding the spin group into the multiplicative group of the Clifford algebra, we take a different approach suggested by the point of view in Howe [75, 77].

This approach obtains the spin representation as a projective representation from the fact that the orthogonal group acts by automorphisms on a ring having a unique representation. The existence of the spin group is a byproduct of the projective representation. This is the same way that the Weil representation is usually constructed from the Stone–von Neumann theorem, with the Clifford algebra replacing the Heisenberg group.

Part IV, we have already mentioned, is largely independent of the earlier parts. Much of it concerned with *correspondences* which were emphasized by Howe, though important examples occur in older work of Frobenius and Schur, Weyl, Weil and others. Following Howe, a *correspondence* is a bijection between a set of representations of a group G with a set of representations of another group H which arise as follows. There is a representation Ω of $G \times H$ with the following property. Let $\pi_i \otimes \pi_i'$ be the irreducible representations of $G \times H$ that occur in the restriction. It is assumed that each occurs with multiplicity one, and moreover, that there are no repetitions among the π_i, and none among the π_i'. This gives a bijection between the representations π_i of G and the representations π_i' of H. Often Ω has an explicit description with special properties that allow us to transfer calculation from one group to the other. Sometimes Ω arises by restriction of a "small" representation of a big group W that contains $G \times H$ as a subgroup.

The first example is the Frobenius–Schur duality. This is the correspondence between the irreducible representations of the symmetric group and the general linear groups. The correspondence comes from decomposing tensor spaces over both groups simultaneously. Another correspondence, for the groups $\mathrm{GL}(n)$ and $\mathrm{GL}(m)$, is embodied in the Cauchy identity. We will focus on these two correspondences, giving examples of how they can be used to transfer calculations from one group to the other.

Frobenius–Schur duality is very often called "Schur–Weyl duality," and indeed Weyl emphasized this theory both in his book on the classical groups and in his book on quantum mechanics. However Weyl was much younger than Schur and did not begin working on Lie groups until the 1920s, while the duality is already mature in Schur's 1901 dissertation. Regarding Frobenius' contribution, Frobenius invented character theory before the relationship between characters and representations was clarified by his student Schur. With great insight Frobenius showed in 1900 that the characters of the symmetric group could be computed using symmetric functions. This very profound idea justifies attaching Frobenius' name with Schur's to this phenomenon. Now Green has pointed out that the 1892 work of Deruyts in invariant theory also contains results almost equivalent to this duality. This came a few years too soon to fully take the point of view of group representation theory. Deruyt's work is prescient but less historically influential than that of Frobenius and Schur since it was overlooked for many years, and in particular Schur was apparently not aware of it. For these reasons we feel the term "Frobenius–Schur duality" is most accurate. See the excellent history of Curtis [39].

Frobenius–Schur duality allows us to simultaneously develop the representation theories of $\mathrm{GL}(n, \mathbb{C})$ and S_k. For $\mathrm{GL}(n, \mathbb{C})$, this means a proof of the Weyl character formula that is independent of the arguments in Part II. For the symmetric group, this means that (following Frobenius) we may use symmetric functions to describe the characters of the irreducible representations of S_k. This gives us a double view of symmetric function theory that sheds light on a great many things. The double view is encoded in the structure of a graded algebra (actually a Hopf algebra) $\mathcal{R}$ whose homogeneous part of degree k consists of the characters of representations of S_k. This is isomorphic to the ring of Λ of symmetric polynomials, and familiarity with this equivalence is the key to understanding a great many things.

One very instructive example of using Frobenius–Schur duality is the computation by Diaconis and Shahshahani of the moments of the traces of unitary matrices. The result has an interesting interpretation in terms of random matrix theory, and it also serves as an example of how the duality can be used: directly computing the moments in question is feasible but leads to a difficult combinatorial problem. Instead, one translates the problem from the unitary group to an equivalent but easier question on the symmetric group.

The $\mathrm{GL}(n) \times \mathrm{GL}(m)$ duality, like the Frobenius–Schur duality, can be used to translate a calculation from one context to another, where it may be easier. As an example, we consider a result of Keating and Snaith, also from random matrix theory, which had significant consequences in understanding the distribution of the values of the Riemann zeta function. The computation in question is that of the $2k$-th moment of the characteristic polynomial of $\mathrm{U}(n)$. Using the duality, it is possible to transfer the computation from $\mathrm{U}(n)$ to $\mathrm{U}(2k)$, where it becomes easy.

Other types of problems that may be handled this way are branching rules: a branching rule describes how an irreducible representation of a group G decomposes into irreducibles when restricted to a subgroup H. We will see instances where one uses a duality to transfer a calculation from one pair (G, H) to another, (G', H'). For example, we may take G and H to be $\mathrm{GL}(p+q)$ and its subgroup $\mathrm{GL}(p) \times \mathrm{GL}(q)$, and G' and H' to be $\mathrm{GL}(n) \times \mathrm{GL}(n)$ and its diagonal subgroup $\mathrm{GL}(n)$.

Chapter 42 shows how the Jacobi–Trudi identity from the representation theory of the symmetric group can be translated using Frobenius–Schur duality to compute minors of Toeplitz matrices. Then we look at involution models for the symmetric group, showing how it is possible to find a set of induced representations whose union contains every irreducible representation exactly once. Translated by Frobenius–Schur duality, this gives some decompositions of symmetric algebras over the symmetric and exterior square representations, a topic that is also treated by a different method in Part II.

Towards the end of Part IV, we discuss several other ways that the graded ring $\mathcal{R}$ occurs. First, the representation theory of the symmetric group has a deformation in the Iwahori Hecke algebra, which is ubiquitous in mathematics, from the representation theory of p-adic groups to the K-theory of flag varieties and developments in mathematical physics related to the Yang–Baxter equation. Second, the Hopf algebra $\mathcal{R}$ has an analog in which the representation theory of $\mathrm{GL}(k)$ (say over a finite field) replaces the representation theory of S_k; the multiplication and comultiplication are parabolic induction and its adjoint (the Jacquet functor). The ground field may be replaced by a p-adic field or an adele ring, and ultimately this "philosophy of cusp forms" leads to the theory of automorphic forms. Thirdly, the ring $\mathcal{R}$ has as a homomorphic image the cohomology rings of flag varieties, leading to the Schubert calculus. These topics are surveyed in the final chapters.

What's New? I felt that the plan of the first edition was a good one, but that substantial improvements were needed. Some material has been removed, and a fair amount of new material has been added. Some old material has been streamlined or rewritten, sometimes extensively. In places what was implicit in the first edition but not explained well is now carefully explained with attention to the underlying principles. There are more exercises. A few chapters are little changed, but the majority have some revisions, so the changes are too numerous to list completely. Highlights in the newly added material include the affine Weyl group, new material about Coxeter groups, Demazure characters, Bruhat order, Schubert and Bott–Samelson varieties, the Borel-Weil theorem the appendix on `Sage`, Clifford algebras, the Keating–Snaith theorem, and more.

Notation. The notations $\mathrm{GL}(n, F)$ and $\mathrm{GL}_n(F)$ are interchangeable for the group of $n \times n$ matrices with coefficients in F. By $\mathrm{Mat}_n(F)$ we denote the ring of $n \times n$ matrices, and $\mathrm{Mat}_{n\times m}(F)$ denotes the vector space of $n \times m$ matrices. In $\mathrm{GL}(n)$, I or I_n denotes the $n \times n$ identity matrix and if g is any matrix, ${}^t g$ denotes its transpose. Omitted entries in a matrix are zero. Thus, for example,

$$\begin{pmatrix} & 1 \\ -1 & \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The identity element of a group is usually denoted 1 but also as I, if the group is $\mathrm{GL}(n)$ (or a subgroup), and occasionally as e when it seemed the other notations could be confusing. The notations $\subset$ and $\subseteq$ are synonymous, but we mostly use $X \subset Y$ if X and Y are known to be unequal, although we make no guarantee that we are completely consistent in this. If X is a finite set, $|X|$ denotes its cardinality.

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