

The Reflection of Life

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The Reflection of Life

Functional Entailment and Imminence
in Relational Biology

 Springer

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Ottawa, Ontario
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ISSN 1574-0463

ISBN 978-1-4614-6927-8

ISBN 978-1-4614-6928-5 (eBook)

DOI 10.1007/978-1-4614-6928-5

Springer New York Heidelberg Dordrecht London

Library of Congress Control Number: 2013932649

Mathematics Subject Classification (2010): 92B99

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To the *rami*[fications] of
my *lignum vitae* and *arbor scientiae*,
my genealogical and academic progenies:

vos palmites estis
ut fructum plurimum adferatis

There are more things in heaven and earth, Horatio,
Than are dreamt of in our philosophy.

— William Shakespeare (c. 1600)
The Tragedie of Hamlet, Prince of Denmarke
Act I, scene v (*First Folio* text, 1623)

Praefatio

Le reflet de la vie

J'ai toujours préféré le reflet de la vie à la vie elle-même.

[I have always preferred the reflection of life to life itself.]

— François Truffaut (1970)
Téléciné, No. 160
(«Spécial Truffaut», mars 1970)

Welcome to the continuation of our exploratory journey in relational biology!
My previous book

 *More Than Life Itself: A Synthetic Continuation in Relational Biology*

was published in 2009. It dealt mainly with the epistemology of life. In its Chapter 13, Ontogenic Vignettes, I briefly mentioned several topics that would be expanded elsewhere, in “my next book”. This monograph you are now reading is that “elsewhere”. It will deal with the ontogeny of life as well as how life evolves from the singular to the plural. This ‘Opus II’ of my epic on relational biology is thus a ‘second image’, hence ‘reflection’.

The roots of the Latin word *reflectere* are *re* ‘back’ and *flectere* ‘to bend’. In geometry, a ‘reflection’ (also spelt ‘reflexion’) is an isometric mapping from a Euclidean space to itself that has a hyperplane as the set of fixed points. When a point is reflected about an axis, for example, the point is ‘bent back’ to a symmetric position on the opposite side of the axis. A reflexive relation ‘bends back’ every element so to be related to itself. In physics, ‘reflection’ is the transition, ‘bending back’, of a wavefront at an interface between two different media so that the wavefront returns into the medium from which it originated. Metaphorically, the word ‘reflection’ can mean ‘turning back one’s thought on some subject’, whence long and careful consideration, an indication, an account, or a description. ‘Reflection’ is a noun of action; it entails plurality. Any object may be the material cause of reflection and be bent back under a formal cause of reflective morphism. The efficient cause of reflection is the interaction of the to-be-reflected entity with its reflector (that which reflects), and the final cause is the

genesis of the reflected output. Common reflected entities are light, heat, sound, and water waves, and—by extension—colour, image, thought, concept, and idea, thence verily exemplified in the sight and sound of *la Nouvelle Vague* that is above all ‘human self-reflection’.

This *liber secundus* of my synthetic continuation in relational biology is, therefore, a ‘reflection’ in every literal and metaphoric sense of the word. Indeed, modelling, the representation of one system in another, is the art that is the ultimate revelatory reflection of life. This is why I have chosen to name this book *The Reflection of Life* (and, for me, the exceedingly *à propos* Truffaut quote clinches it). I nominate it thus, despite being fully aware that the title is somewhat generic and formulaic: the shelf of books entitled *The Y of X* is quite crowded. (Incidentally, *The Origin of Species* is not a fitting example here. Although this arguably most famous scientific publication is often referred to by this more declarative name, Charles Darwin’s original 1859 title was the verbose *On the Origin of Species by Means of Natural Selection, or the Preservation of Favoured Races in the Struggle for Life*.) Even in my subject area of mathematical biology, the name *The Y of Life* is well represented; among them are, for example, Denis Noble’s 2006 *The Music of Life* and Ian Stewart’s 2011 *The Mathematics of Life* (both, I may add, excellent books). My rather specific subtitle for the book should, nevertheless, serve to distinguish it: I am reasonably certain (in the strong-limit sense of almost sure convergence), an infinitude of typing monkeys notwithstanding, that the very sequence of words *Functional Entailment and Imminence in Relational Biology* has not appeared in print elsewhere.

A main theorem in relational biology says:

*A natural system is an organism
if and only if it is closed to efficient causation.*

If such a central issue of what life is can be so succinctly defined, then why is relational biology not as well known as it deserves to be? It may be because category theory, the *lingua franca* of relational biology, is not a very accessible branch of mathematics; it is not uncommon for a university student graduating in mathematics not to have taken a course on the subject. It may also be true that many in the rest of the community of biologists at large were antagonistic towards the Rashevsky-Rosen school, perhaps not so much on petty personal(ity) conflicts than on points of philosophical difference.

We are not denying that an underlying material basis is needed and that *some* information on living systems may derive from their material bases. The real *nature* of living systems, however, is not conveyed by their material basis. Physicochemical structures do not dictate functions; physicochemical structures are manifestations of functions.

Many biologists are convinced that “biology is inherently messy”, and some aggrandizers have even presumptuously spoken for all and proclaimed as a “conviction” of biologists that the actual complex behaviour of real organisms would be lost in simple even if elegant idealizations. They regard cells and organisms as machine-like systems, a metaphor that even today dominates biology. Even for those biologists that are not as blatantly reductionistic, they would still

brand relational models “(over-)simplifications”, and advocate (and advertise) the euphemistic “biologically realistic models” or “models of biological relevance”. But what do “realistic” and “relevant” imply? Do they not implicitly remain the insistence that everything in biology must be explainable in terms of the underlying physicochemical materials? Contrariwise, from the standpoint of relational biology, machine-like systems are in fact simple; biological systems are complex precisely because their essence is lost when modelled as machines.

I may conjecture that this physicochemical bias has puritanical roots. Let me state that I am not referring to (capitalized) Puritanism that is the theological creed and social vision, but only to a debased, secularized, conservative form of (lower-case) puritanism, that of “anguished self-flagellation” and “suffering is purposeful”. To wit, the slogan of many experimental biologists is that “real biologists” must “get their hands dirty”, and that they must keep their “feet on the ground” (extolled from their *pieds-à-terre* in ivory towers; cf. [Rosen 2006] for an anecdote)! It is not that they do not appreciate that nature *itself* is beautiful; it is just that they feel the worthiness of an experimenter’s *study* of nature ought somehow to be linked to the degree of messiness and dirtiness of the endeavour.

I wonder how people can appreciate the ontological beauty of nature but then insist on its epistemological ugliness.

Function dictates structure: relational biology begins with mathematical ideas and seeks realizations in natural systems. The Book of Nature is written in the language of mathematics. A theorist’s conception of nature is based on *beauty*. I shall let G. H. Hardy, pure mathematician *par excellence*, have the last word:

The mathematician’s patterns, like the painter’s or the poet’s, must be *beautiful*; the ideas, like the colours or the words, must fit together in a harmonious way. Beauty is the first test: there is no permanent place in the world for ugly mathematics.

— G. H. Hardy (1940)
A Mathematician’s Apology
 § 10



A. H. Louie
 19 May, 2012

Nota bene

Prerequisites

The cast and crew of mathematical and biological characters in ‘Opus I’, my previous book *More Than Life Itself* [Louie 2009], include partially ordered sets, lattices, simulations, models, Aristotle’s four causes, graphs, categories, simple and complex systems, anticipatory systems, and metabolism-repair [(M,R)-] systems. In this ‘Opus II’, my present book *The Reflection of Life*, I shall expand the cast and crew to employ set-valued mappings, adjacency matrices, random graphs, and interacting entailment networks. If the theme of Opus I is *one* (M,R)-system, then the theme of Opus II is *two* interacting (M,R)-systems.

Throughout this book I shall adopt the notation and terminology and draw upon results from *More Than Life Itself*. Since I shall be referring to that book many times, henceforth the canonical symbol *ML* will be used in its stead. In this present volume, when various topics are encountered, when appropriate I shall refer the reader to relevant passages in *ML* for further exploration; the notation ‘*ML*: m.n’ refers to Section m.n (in Chapter m) of *ML*.

I assume the reader is already familiar with the premises of the Rashevsky-Rosen school of relational biology, as explicated in *ML*. In particular, I recursively enlist all the assumptions made in the Nota bene of *ML* (pp. xxiii–xxiv) and include them as prerequisites for continuing our journey in relational biology. The Exordium that follows next is a terse introduction to relational biology, but it is a précis, and not a substitute of the in-depth exploration of the subject contained in *ML*.

As prerequisites, the reader should have already understood the following statements.

Definition (*ML*: 5.15) The entailment of an efficient cause is called *functional entailment*.

Definition (*ML*: 6.23) A natural system is *closed to efficient causation* if its every efficient cause is (functionally) entailed within the system.

Postulate of Life (*ML*: 11.28) A natural system is an *organism* if and only if it realizes an (M,R)-system.

Theorem (*ML*: 11.29) *A natural system is an organism if and only if it is closed to efficient causation.*

This sequence of statements is a succinct summary of our answer to the “What Is Life?” question. Life is a phenomenon that sets organisms apart from nonliving systems and dead organisms, and life is manifested through the relations among the processes of metabolism (M) and repair (R). It is through a network of efficient causes that an (M,R)-system models a living system (i.e. ‘organism’ in its most general sense), so a reductionistic model based strictly on material causation does not qualify.

The defining characteristic of a *living system*, ‘closure to efficient causation’, anchors on the key concept of *functional entailment*. (Robert Rosen coined the term in Section 5I of his masterwork *Life Itself* [Rosen 1991].) Note that an efficient cause that is entailed is ‘function’ in both its mathematical sense (‘mapping’) and its biological sense (‘a mode of action by which a thing fulfils its purpose’; *ML*: 0.28). The *imminence* (which I shall define in this book, in Section 7.16) of a mapping f is the collection of all the (functionally) f -entailed entities that can *themselves* entail. *Functional entailment* and *imminence*, the ‘local’ and ‘global’ manifestations of the concept, play leading roles in this Opus II of my epic on relational biology; thus the subtitle.

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Exordium

An Introduction to Relational Biology

My 2009 book *ML* has garnered some attention and has engendered/sustained/renewed interest on the subject of relational biology. The journal *Axiomathes* (the theme of which is ‘Where Science Meets Philosophy’) dedicated a recent issue (volume 21 number 3, September 2011; [Poli 2011]) to discussing the nuances of *ML*. Entitled ‘Essays on *More Than Life Itself*’, the special topical issue comprises four essays commenting on *ML* and my responses [Louie 2011] to these comments. The growing interest also led to my being invited to conferences to speak on the subject. This Exordium is a representation of one of these lectures. It is included herein as a review, or a ‘refresher of the whys and wherefores’, as it were, of concepts considered in detail in *ML*.

E.1 The Interrogative Science is an activity based on the interrogative: one poses questions about nature and attempts to gain knowledge by answering these questions.

Aristotle contended that one did not really know a ‘thing’ (which to Aristotle meant a natural system) until one had answered its ‘why?’ with its *αἴτιον* (primary or original ‘cause’). In other words, Aristotle’s *science* is precisely the subjects for which one seeks the *αἴτια* to the interrogative ‘?’.

Aristotle’s original Greek term *αἴτιον* (*aition*) was translated into the Latin *causa*, a word which might have been appropriate initially, but which had unfortunately diverged into our contemporary notion of ‘cause’, as ‘that which produces an effect’ (more on this shortly). The possible semantic equivocation may be avoided if one understands that Aristotle’s original idea had more to do with ‘grounds or forms of explanation’, so a more appropriate Latin rendering, in retrospect, would probably have been *explanatio*.

E.2 What Is Life? Biology is the study of life. The ultimate biological question is, then, “What is life?”

This was the question Erwin Schrödinger posed in 1943 and attempted to answer in a series of lectures delivered in Dublin; the corresponding book was published in 1944 [Schrödinger 1944]. With decades of hindsight and further advances in biology, parts of the book may now appear dated. But the originality

expressed in this book is not diminished, and the fact that it is still in print is a testimony to its continuing significance.

The Schrödinger question “What is life?” is an abbreviation. A more explicitly posed expansion is

“What distinguishes a living system from a non-living one?”

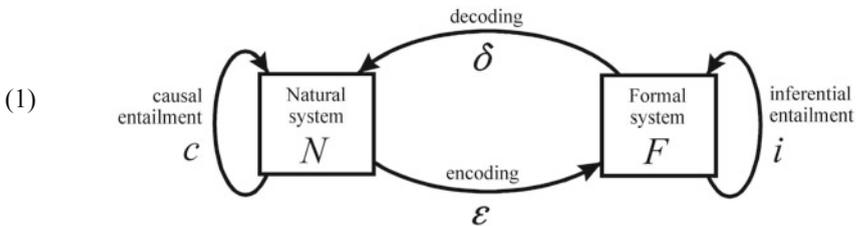
alternatively,

“What are the defining characteristics of a natural system
for us to perceive it as being alive?”

These are epistemological forms of the question.

E.3 The Modelling Relation *Causality* in the modern sense, the principle that every effect has a cause, is a reflection of the belief that successions of events in the world are governed by definite relations. *Natural Law* posits the existence of these *entailment* relations *and* that this causal order can be *imaged* by implicative order.

A *modelling relation* is a commutative functorial encoding and decoding between two systems. Between a natural system (an object partitioned from the physical universe) N and a formal system (an object in the universe of mathematics) F , the situation may be represented in the following canonical diagram:



The encoding ε maps the natural system N and its causal entailment c therein to the formal system F and its internal inferential entailment i ; that is,

(2)
$$\varepsilon : N \rightarrow F \text{ and } \varepsilon : c \rightarrow i .$$

The decoding δ does the reverse. The entailments satisfy the commutativity condition

(3)
$$c = \varepsilon \triangleright i \triangleright \delta .$$

(Stated graphically, equality (3) says that, in diagram (1), tracing through arrow c is the same as tracing through the three arrows ε , i , and δ in succession.) Thence related, F is a *model* of N , and N is a *realization* of F . In terms of the modelling relation, then, Natural Law is a statement on the existence of causal entailment c and the encodings $\varepsilon: N \rightarrow F$ and $\varepsilon: c \rightarrow i$.

A formal system may simply be considered as a *set* with additional mathematical structures. So the mathematical statement $\varepsilon: N \rightarrow F$, that is, the posited existence for every natural system N a model formal system F , may be stated as the axiom

Everything is a set.

A *mapping* is an inference that assigns to each element of one set a unique element of another set. In elementary mathematics, when the two sets involved are sets of numbers, the inference process is often called a *function*. So ‘mapping’ may be considered a generalization of the term, when the sets are not necessarily of numbers. (The use of ‘mapping’ here avoids semantic equivocation and leaves ‘function’ to its biological meaning.)

Causal entailment in a natural system is a network of interacting processes. The mathematical statement $\varepsilon: c \rightarrow i$, that is, the functorial correspondence [ML: A.10] between causality c in the natural domain and inference i in the formal domain, may thus be stated as an epistemological principle, the axiom

Every process is a mapping.

Together, the two axioms are the mathematical formulation of Natural Law. These self-evident truths serve to explain “the unreasonable effectiveness of mathematics in the natural sciences”.

E.4 Biology Extends Physics A living system is a material system, so its study shares the material cause with physics and chemistry. Reductionists claim this, therefore, makes biology reducible to ‘physics’. *Physics*, in its original meaning of the Greek word *φύσις*, is simply (the study of) *nature*. So in this sense, it is tautological that everything is reducible to physics. But the hardcore reductionists, unfortunately, take the term ‘physics’ to pretentiously mean ‘(the toolbox of) *contemporary physics*’.

Contemporary physics that is the physics of mechanisms reduces biology to an exercise in molecular dynamics. This reductionistic exercise, for example, practised in biochemistry and molecular biology, is useful and has enjoyed popular success and increased our understanding life by parts. But it has become evident that there are incomparably more aspects of natural systems that the physics of mechanisms is *not* equipped to explain.

Biology is a subject concerned with organization of relations. Physicochemical theories are only surrogates of biological theories, because the manners in which the shared matter is organized are fundamentally different.

Hence, the behaviours of the realizations of these mechanistic surrogates are different from those of living systems. This in-kind difference is the impermeable dichotomy between *predicativity* and *impredicativity*. (I shall explicate these two antonyms presently.)

In his 1944 book, Schrödinger wrote:

“... living matter, while not eluding the ‘laws of physics’ as established up to date, is likely to involve ‘other laws of physics’ hitherto unknown, which however, once they have been revealed, will form just as integral a part of science as the former.”

There have, of course, been many interpretations of what these ‘other laws of physics’ might have been. Schrödinger himself likely thought of extensions in thermodynamical terms. It is, however, nothing new in the history of physics that ‘other laws of physics’ have been added to the repertoire from time to time when ‘the toolbox of contemporary physics’ became inadequate. The mathematical toolbox of calculus was sufficient for Newtonian mechanics. Tensor geometry had to be recruited for relativity. Operator theory was the appropriate mathematical language of quantum physics. I contend that biology extends physics, and to accordingly expand the toolbox, one needs to enlist *category theory*.

Any question becomes unanswerable if one does not permit oneself a large enough universe to deal with the question. The failure of presumptuous reductionism is that of the inability of a small surrogate universe to exhaust the real one. Equivocations create artefacts. The limits of mechanistic dogma are very examples of the restrictiveness of self-imposed methodologies that fabricate non-existent artificial ‘limitations’ on science and knowledge. The limitations are due to the nongenericity of the methods and their associated bounded microcosms. One learns something new and fundamental about the universe when it refuses to be exhausted by a posited method.

E.5 Relational Biology The study of biology from the standpoint of ‘organization of relations’ is a subject called *relational biology*. It was founded by Nicolas Rashevsky (1899–1972) in the 1950s, thence continued and flourished under his student Robert Rosen (1934–1998), my PhD supervisor.

The essence of reductionism in biology is to keep the matter of which an organism is made, and throw away the organization, with the belief that, since physicochemical *structure implies function*, the organization can be effectively reconstituted from the analytic material parts.

Relational biology, on the other hand, keeps the organization and throws away the matter; *function dictates structure*, whence material aspects are entailed.

In terms of the modelling relation, reductionistic biology is physicochemical process seeking models, while relational biology is organization seeking realizations. Stated otherwise, reductionistic biology begins with the material system and relational biology begins with the mathematics. Thus, the principles of relational biology may be considered the operational inverse of (and complementary to) reductionistic ideas. It must be emphasized that both

approaches are valuable, each answering questions that the other is not equipped to answer. ‘Structure implies function’ has beneficial epistemological implications, while ‘function dictates structure’ better addresses ontological issues. What renders hardcore reductionism a falsehood is their practitioners’ overreaching claim of genericity, their indignant exclusion of other approaches (which they presumptuously consider to be illegitimate), and their self-declared exclusive ownership of objectivity besides. One world is not enough.

In the relational-biological approach, the answer to our “What is life?” question will define an organism as a material system that realizes a certain kind of relational pattern, whatever the particular material basis of that realization may be. For the remainder of this exposition, I shall proceed to answer this question and use the process of reaching this goal to illustrate the methods of relational biology.

E.6 Mapping and Its Relational Diagram In relational biology, we begin with a formal system, with biology entailed as its realization. So let me begin with a mathematical object, a *mapping* f from set A to set B . It is commonly denoted thus:

$$(4) \quad f: A \rightarrow B.$$

The mapping (4) may alternatively be represented in its category-theoretic notation

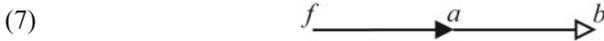
$$(5) \quad f \in H(A, B),$$

where $H(A, B)$ denotes a set of mappings from set A to set B and is called a *hom-set*. Essentially, (5) says that $H(A, B)$ is a collection of mappings from set A to set B , and f , being a member of this collection, is one such mapping.

Another way to represent the mapping (4) is its ‘element-chasing’ version: if $a \in A$, $b \in B$, and the variables are related as $b = f(a)$, then one may use the ‘maps to’ arrow (note the short vertical line segment at the tail of the arrow) and write

$$(6) \quad f: a \mapsto b.$$

Let me introduce a final representation of the mapping f , its *relational diagram in graph-theoretic form*. It may be drawn as a network with three *nodes* and two *directed edges*, that is, a directed graph (or *digraph* for short):



This graph-theoretic representation allows a ready identification of components of a mapping with the four Aristotelian causes that respond to the interrogative “Why mapping?”.

The input $a \in A$ is the *material cause*. The output $b \in B$ is the *final cause*. The *hollow-headed arrow* denotes the *flow* from input $a \in A$ to output $b \in B$, whence the final cause of the mapping may be identified also as the hollow-headed arrow that terminates on the output:



The *efficient cause* is the *function* of the mapping f as a *processor*; thus, it may be identified as f itself. The *solid-headed arrow* denotes the induction of or constraint upon the flow by the processor f , whence the efficient cause of the mapping may be identified also as the solid-headed arrow that originates from the processor:



The *formal cause* of the mapping is the ordered pair of arrows:



that is, the ordered pair of \langle processor, flow \rangle .

E.7 Efficient Cause Since the efficient cause will turn out to be the crucial *aition* in relational biology, I shall explicate it further. Aristotle’s *κινητικός* (*kinetikos*) is rendered into *efficare* in Latin: the efficient cause is “one who puts in motion, that which brings the thing into being, the source of change, that which makes what is made, the ‘production rule’”. Note that efficient cause in the Aristotelian sense is simply ‘the processor’, and the adjective ‘efficient’ has nothing to do with its common-usage sense that is ‘productive with minimum waste or effort’.

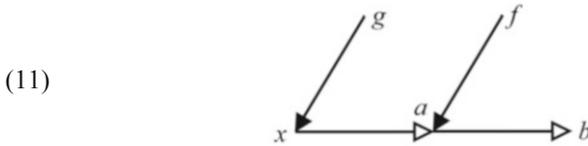
The Natural Law axiom “Every process is a mapping.” encodes natural processes into mappings; in particular, the encoding identifies an efficient cause of

a natural process with the efficient cause of the corresponding mapping. The isomorphic correspondence between the *solid-headed arrow* (9) and the efficient cause of a mapping then completes the linkage in our formalism. Each statement on entailment thus has three analogous formulations, concerning:

- i. Causal entailment patterns among efficient causes of natural processes
- ii. Inferential entailment paths among efficient causes of mappings
- iii. Graphical entailment networks among solid-headed arrows

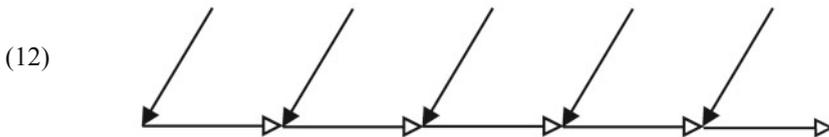
E.8 Compositions The relational diagrams of mappings may *interact*: two mappings, with the appropriate domains and codomains, may be connected at different common nodes.

As a first example, consider $g : x \mapsto a$ and $f : a \mapsto b$; thus, *the output of g is the input of f* (the common ‘middle’ element a). In terms of hom-sets, one has $g \in H(X, A)$ and $f \in H(A, B)$ (where, naturally, $x \in X$, $a \in A$, and $b \in B$); thus, *the codomain of g is the domain of f* (the common ‘middle’ set A). The relational diagrams of these two mappings connect at the common node a as



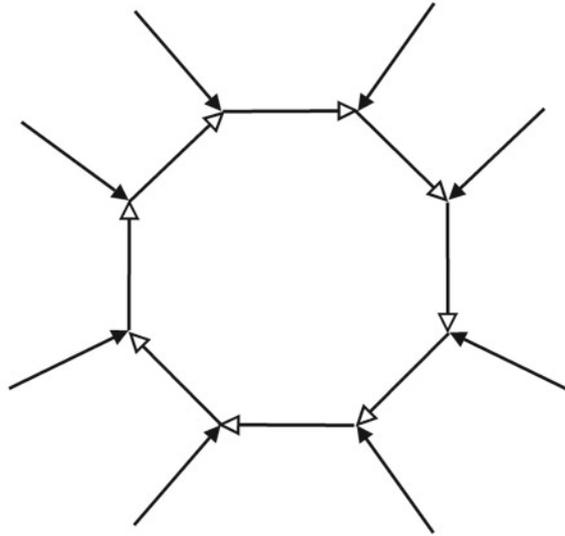
This *sequential composition* of relational diagrams represents the composite mapping $f \circ g \in H(X, B)$ with $f \circ g : x \mapsto b$.

When several mappings are linked by sequential compositions, one has a *sequential chain*:



When the first and last mappings in a sequential chain are themselves linked by sequential composition, the chain folds up into a *sequential cycle*:

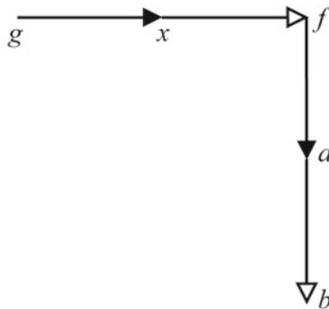
(13)



Note that *within* a sequential cycle, the arrows involved have a consistent direction and are *all hollow-headed* (with solid-headed arrows *peripheral* to the cycle). That is, the compositions involved in the closed path are all sequential, and each final cause has the additional role of being the material cause of the subsequent mapping. A sequential cycle may, therefore, be called a *closed path of material causation*.

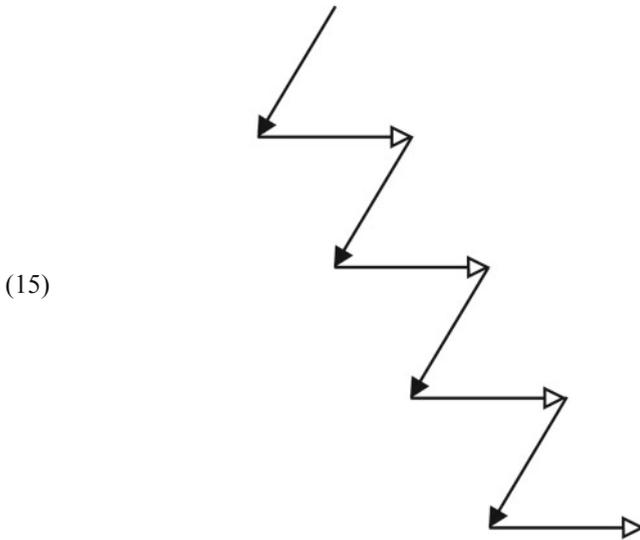
Next, consider two mappings g and f with $g : x \mapsto f$ and $f : a \mapsto b$ —now *the output of g is itself the mapping f* . The hom-sets involved are $g \in H(X, H(A, B))$ and $f \in H(A, B)$: thus, *the codomain of g contains f* . Because of this ‘containment’, the mapping g may be considered to occupy a higher ‘hierarchical level’ than the mapping f (and that the hom-set $H(X, H(A, B))$ is at a higher hierarchical level than $H(A, B)$). For these two mappings, one has the *hierarchical composition* of relational diagrams:

(14)

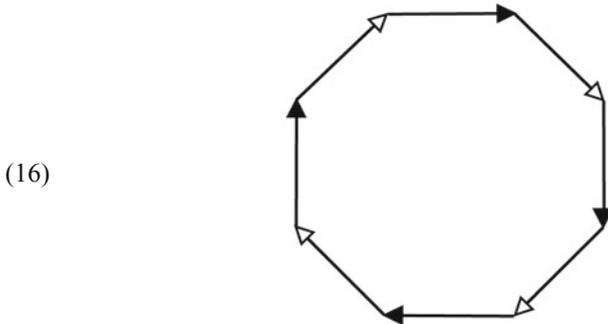


Since the final cause (i.e. output) of g is the efficient cause of f , the mapping g may be considered an ‘efficient cause of efficient cause’. An iteration of efficient causes is inherently hierarchical, in the sense that a lower-level efficient cause is contained within a higher-level efficient cause. In sequential composition, the first mapping g produces something to be operated on, but in hierarchical composition, the first mapping g produces instead an operator itself. Hierarchical composition thus concerns a ‘different’ mode of entailment, which is given the name of *functional entailment*.

Similar to sequential compositions, hierarchical compositions may form a *hierarchical chain*:



and a *hierarchical cycle*:



Note that, in contrast to a sequential cycle (13), *solid-headed arrows* (along with hollow-headed arrows) are definitive components of a hierarchical cycle. Efficient causes are relayed; thus, a hierarchical cycle is a *closed path of efficient causation*.

E.9 Impredicativity In logic, the *predicate* is what is said or asserted about an object. It can take the role as either a property or a relation between entities. Thus, *predicate calculus* is the type of symbolic logic that takes into account the contents (i.e. predicate) of a statement. The defining property $p(x)$ of a subset P in the universe U , as in

$$(17) \quad P = \{x \in U : p(x)\},$$

is an example of a predicate, since it *asserts unambiguously* the property that x must have in order to belong to the set P .

Contrariwise, a definition of an object is said to be *impredicative* if it invokes (mentions or quantifies over) the object itself being defined, or perhaps another set which contains the object being defined. In other words, *impredicativity* is the property of a *self-referencing definition* and may *entail ambiguities*. An impredicative definition often appears circular, as what is defined participates in its own definition.

Impredicative definitions usually cannot be bypassed and are mostly harmless. But there are some that lead to paradoxes. The most famous of a problematic impredicative construction is Russell's paradox, which involves the set of all sets that do not contain themselves:

$$(18) \quad \{x : x \notin x\}.$$

(This foundational difficulty is only avoided by the restriction to a naive set-theoretic universe that explicitly prohibits self-referencing constructions.)

It is evident that a hierarchical cycle, with its cyclic collection of mutually entailing efficient causes, is impredicative. In other words, a hierarchical cycle is an *impredicative cycle of inferential entailment*. A closed path of efficient causation must form a hierarchical cycle of containment: both the hierarchy of containment and the cycle are essential attributes of this closure.

Through the encoding that identifies an efficient cause of a natural process with the efficient cause of the corresponding mapping, one may conclude that

*A natural system has a model containing a hierarchical cycle
if and only if it has a closed path of efficient causation.*

Stated otherwise, a hierarchical cycle is the relational diagram in graph-theoretic form of a closed path of efficient causation.

E.10 Nonsimulability An *algorithm* is a computation procedure that requires in its application a *rigid stepwise mechanical execution of explicitly stated rules*. It is presented as a prescription, consisting of a finite number of instructions. It halts after a finite number of steps. It has no room for ambiguity.

Predicates are algorithmic. Impredicativity is everything that an algorithm is *not*.

A mapping is *simulable* if it is definable by an algorithm. A formal system, an object in the universe of mathematics, may be considered a collection of mappings connected by the system's entailment pattern (i.e. its graph, which may itself be considered a mapping). So by extension, a formal system is *simulable* if its entailment pattern and all of its mappings are simulable. Simulability entails finiteness: that the corresponding Turing machine halts after a *finite* number of steps, that the corresponding algorithmic process is of *finite* length, and that the corresponding program is of *finite* length.

Impredicativity has many consequences. In view of its being the antithesis of things algorithmic, one of these consequences is, therefore, nonsimulability.

Among the entailment networks (12), (13), (15), and (16) that we have considered, the first three, namely, sequential chain, sequential cycle, and hierarchical chain, are simulable, but the last one, hierarchical cycle, is not. The nonsimulability of a hierarchical cycle has been proven using lattice theory. I state this theorem formally as

A formal system that contains a hierarchical cycle is not simulable.

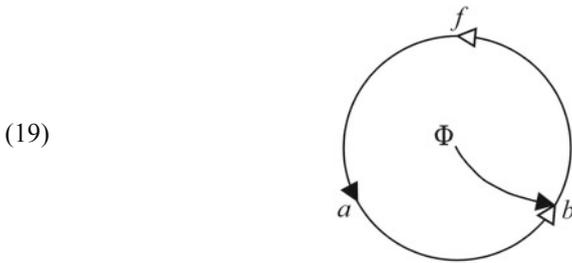
For natural systems, a *deadlock* is a situation wherein competing actions are waiting for one another to finish, and thus none ever does. A set of processes is in a deadlock state when every process in the set is waiting for an event that can be *caused* only by another process in the set. This is a realization, a relational analogue, of impredicativity. In computer science, deadlock refers to a specific condition when two or more processes are each waiting for another to release a resource, or more than two processes are waiting for resources in a circular chain. Implementation of hierarchical cycles (or attempts to execute ambiguous codes in general) will lead a program to either a deadlock or an endless loop. In either case, the program does not terminate. This is practical verification that a hierarchical cycle is *not* simulable.

E.11 Biological Realization: Metabolism-Repair System Every process is a mapping. The crucial biological process of *metabolism* may, therefore, be represented as a mapping $f : a \mapsto b$ (equivalently, $f \in H(A, B)$); an *enzyme* may be the realization of the efficient cause f , with material input and output metabolites realizations of a and b . Networks of mappings in sequential composition are, then, models of metabolic pathways.

Some biochemical processes produce enzymes as outputs. Such a process may naturally be modelled as a mapping of the form $\Phi : x \mapsto f$ (equivalently, $\Phi \in H(X, H(A, B))$). The morphism Φ may be considered *repair*: its codomain

is $H(A,B)$, so it is a mapping that creates new copies of enzymes f , hence a *gene* that ‘repairs’ (or replenishes) the metabolism process. The repair map Φ and the metabolism map f are thus in hierarchical composition.

A typical eukaryotic cell is compartmentalized into two observably different regions, the cytoplasm and the nucleus. Metabolic activities mainly occur in the cytoplasm, while repair processors (i.e. genes) are contained in the nucleus. Repair in cells generally takes the form of a continual synthesis of basic units of metabolic processor (i.e. enzymes), using as inputs materials provided by the metabolic activities themselves. In particular, the simplest domain of the repair map Φ may be the codomain of metabolism f , the latter’s ‘output set’ B (i.e. $\Phi: b \mapsto f, \Phi \in H(B, H(A, B))$), whence metabolism and repair combine into the relational diagram



This geometry gives a graphic representation of the metabolism component as the abstract equivalent of ‘cytoplasm’ and the repair component as the abstract counterpart of ‘nucleus’.

What if the repair components themselves need repairing? New mappings representing *replication* (serving to replenish the repair components) may be defined. A replication map must have as its codomain the hom-set $H(X, H(A, B))$ to which repair mappings Φ belong, so it must be of the form

$$(20) \quad \beta: Y \rightarrow H(X, H(A, B))$$

for some set Y (where Y contains ingredients already present in the cell). In the simplest case, when $X = B$, one may choose $Y = H(A, B)$; so (20) becomes

$$(21) \quad \beta: H(A, B) \rightarrow H(B, H(A, B)).$$

It turns out that under stringent but not prohibitively strong conditions, the replication mapping β may already be entailed within the components present. There are many ways in which this happens; one natural way is that an isomorphic

correspondence may be defined between b and β , whence the mapping (21) may be equivalently represented as

$$(22) \quad b: f \mapsto \Phi.$$

The relational diagram of the entailment among the metabolism-repair-replication mappings

$$(23) \quad \{ f : a \mapsto b, \Phi : b \mapsto f, b : f \mapsto \Phi \}$$

is then

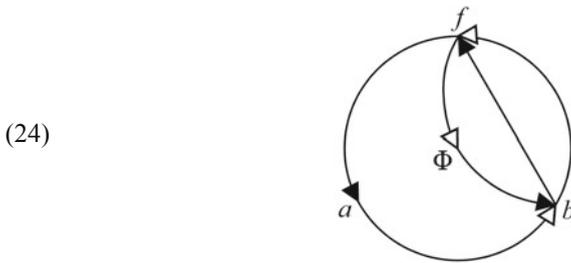
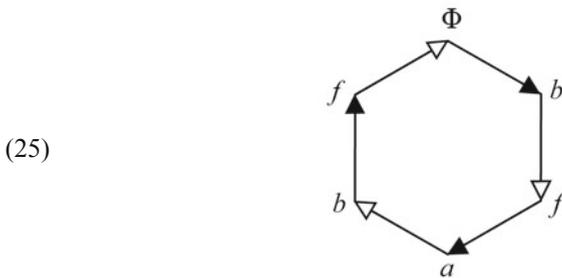


Diagram (24) is the relational diagram in graph-theoretic form of the simplest *metabolism-repair system* (or (M,R) -system for short), introduced by Robert Rosen in the late 1950s.

Note that (24) is a hierarchical cycle. The entailment pattern is more evident when the relational diagram is unfolded thus:



One may also note that there is no ‘privileged’ position of any of the three mappings involved. They are in cyclic entailment and may be assigned the labels of metabolism, repair, and replication in any cyclic permutation. The all-

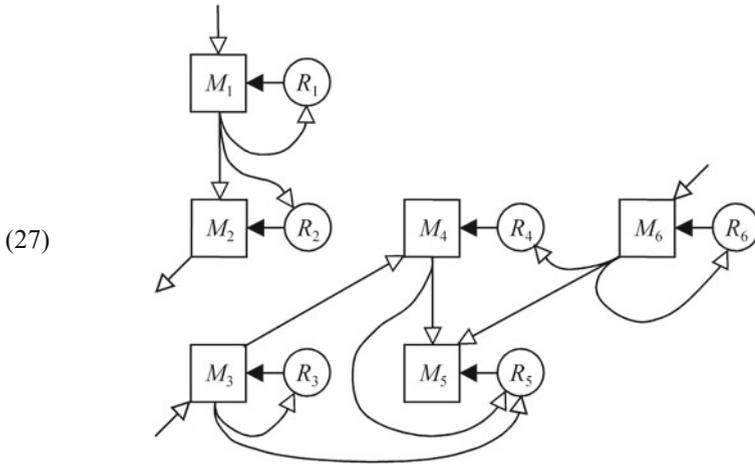
important feature is that the mappings form a hierarchical cycle; stated otherwise, the simplest (M,R)-system is a hierarchical-cycle model of a cell.

In the specialization of the replication map β from (20) to (21), many simplifying assumptions have been made to create the three-mapping $\{f, \Phi, b\}$ hierarchical cycle. A more sophisticated (M,R)-system model of a cell would contain a large number of metabolism and repair components connected in a complex entailment network, since in a cell there are obviously many more than three interacting processes. (Diagram (24) actually already captures the essence of all (M,R)-systems, and indeed it is possible in principle to reduce every abstract (M,R)-system to this simple form by making the three mappings involved sufficiently complex. One must, nevertheless, not lose sight of the network aspect of (M,R)-systems.)

Metabolism may alternatively be considered an input-output system, with the mapping f representing the transfer function of the 'block', the domain A as the set of inputs, and the codomain B as the set of outputs. Similarly, *repair* may be considered an input-output system, with the mapping Φ representing the transfer function of the block, the domain B as the set of inputs, and the codomain $H(A,B)$ as the set of outputs. With the addition of entailment arrows for environmental inputs and outputs, and the abbreviated representation by the symbols M and R of the components, the relational diagram (19) may be represented as this simple network of one metabolism component and one repair component:



In general, a metabolism-repair network consists of many metabolism and repair components, with the requisite connections that the outputs of a repair component are observables in the hom-set of its corresponding metabolism component; the metabolism components may be connected among themselves by their inputs and outputs; and repair components must receive at least one input from the outputs of the metabolism components of the network. The following is a sample (M,R)-network (still relatively simple) with six pairs of metabolism-repair components:



One may easily visualize larger (M,R)-networks with thousands of components.

E.12 Closure to Efficient Causation Suppose a natural system *contains* a closed path of efficient causation, then *some* of its efficient causes are in cyclic entailment of one another. Their corresponding mappings must then form a hierarchical cycle. If it so happens that *all* of a natural system’s efficient causes entail one another, then it must have a model in which *all* solid-headed arrows are components of hierarchical cycles (e.g. diagram (24) of the simplest (M,R)-system). Having *all* efficient causes entailed within the system is a more stringent requirement than having just *some*, and members of this subset of natural systems are given a special description: *closed to efficient causation*.

A natural system is *closed to efficient causation* if its every efficient cause is entailed within the system.

The correspondence between an efficient cause and a solid-headed arrow implies:

A natural system is *closed to efficient causation* if and only if each connected component in its relational diagram has a closed path that contains all the solid-headed arrows.

I mention in passing that “a closed path that contains all the solid-headed arrows” is related to the concept of *traversability* (one continuous trace of the edges in a graph, passing along each edge exactly once) in network topology. Thus, the study of ‘closed to efficient causation’ can make use of the powerful results from the mathematical theory of topology (in addition to lattice theory and category theory that we have already encountered).

Not all metabolism-repair networks satisfy the stringent requirements for entailment closure. The defining characteristic of an (M,R) -system that makes it a model of cells is the self-sufficiency in the networks of metabolism and repair components, in the sense that every mapping is entailed within, in short, closure to efficient causation.

The answer to our “What is life?” question according to the Rashevsky-Rosen school of relational biology, in a nutshell, is that an *organism*—the term is used in the sense of an ‘autonomous life form’, that is, any living system (including, in particular, cells)—admits a certain kind of relational description, that it is ‘closed to efficient causation’. Explicitly:

*A material system is an organism
if and only if it is closed to efficient causation.*

This ‘self-sufficiency’ in efficient causation is what we implicitly recognize as the one feature that distinguishes a living system from a nonliving one.

In terms of (M,R) -systems, we may state the **Postulate of Life**:

*A natural system is an organism
if and only if it realizes an (M,R) -system.*

Thus, an (M,R) -system is the very model of life, and, conversely, life is the very realization of an (M,R) -system.

Prolegomenon

Cardinalis

Not everything that counts can be counted, and not everything
that can be counted counts.

— attributed to Albert Einstein

As I did in *ML*, in this book, I assume that the reader is familiar with the basic facts of *naive set theory*, as presented, for example, in Halmos [1960]. In this prologue, however, I shall present some set-theoretic and logical preliminaries; this is more for the clarity of notations (especially for those non-standardized ones) than for the concepts themselves.

Sets

0.1 Definition If A and B are sets and if every element of A is an element of B , then A is a *subset* of B , denoted

$$(1) \quad A \subset B.$$

Note that this symbolism of containment means *either* $A = B$ (which means the sets A and B have the same elements; *ML*: 0.2: Axiom of Extension) *or* A is a proper subset of B (which means that B contains at least one element that is not in A). Two sets A and B are equal if and only if $A \subset B$ and $B \subset A$ (*ML*: 0.4).

0.2 Definition If X is a set, the *power set* $\mathfrak{P}X$ of X is the family of all subsets of X .

An alternate notation of the power set $\mathfrak{P}X$ is 2^X (*cf.* *ML*: A.3 for the etymology).

0.3 Definition The *relative complement* of a set A in a set B is the set of elements in B but not in A :

$$(2) \quad B \sim A = \{x \in B : x \notin A\}.$$

When B is the ‘universal set’ U (of some appropriate universe under study, e.g. the set \mathbf{N} of all natural systems), the set $U \sim A$ is denoted A^c , that is,

$$(3) \quad A^c = \{x \in U : x \notin A\},$$

and is called simply the *complement* of the set A .

0.4 Number Sets Various sets of numbers are denoted thus:

- i. *Natural numbers* (‘positive integers’) $\mathbf{N} = \{1, 2, 3, 4, \dots\}$
- ii. *Whole numbers* (‘nonnegative integers’) $\mathbf{N}_0 = \{0\} \cup \mathbf{N} = \{0, 1, 2, 3, 4, \dots\}$
- iii. *Integers* $\mathbf{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$
- iv. *Rational numbers* (‘fractions’) $\mathbf{Q} = \left\{ \frac{p}{q} : p \in \mathbf{Z}, q \in \mathbf{N} \right\}$
- v. *Real numbers* \mathbf{R}
- vi. *Complex numbers* \mathbf{C}

The six number sets are related by

$$(4) \quad \mathbf{N} \subset \mathbf{N}_0 \subset \mathbf{Z} \subset \mathbf{Q} \subset \mathbf{R} \subset \mathbf{C}$$

in which all containments are proper.

Equipotence

0.5 Definition Two sets are *equipotent* (to each other) if there exists a bijective mapping, that is, a one-to-one correspondence, between them (*cf.* *ML*: 1.8).

Stated otherwise, two sets are equipotent if they are isomorphic in the category **Set** (*cf.* *ML*: A.6). It is evident that equipotence is an equivalence relation (*ML*: 1.11). The symmetry of the relation also allows the usage ‘set A is equipotent to set B ’, since it implies ‘set B is equipotent to set A ’, whence A and B are equipotent to each other. One also occasionally sees the usage of ‘equipollent’, or even ‘equinumerous’, for the same concept.

0.6 Schröder-Bernstein Theorem *If each of two sets is equipotent to a subset of the other, then the two sets are equipotent.*

Since every set itself is its own subset, the converse of the Schröder-Bernstein Theorem, that if two sets are equipotent then each is equipotent to a subset of the other, is trivially true.

0.7 Law of Trichotomy of Equipotence *Two sets are either equipotent to each other, or one is equipotent to a subset of the other.*

If two sets are equipotent, then it is easy to see that their power sets are equipotent. But a set is never equipotent to its own power set; this is

0.8 Cantor's Theorem *Every set is equipotent to a proper subset of its power set, but is not equipotent to the power set itself.*

Cardinality

0.9 Definition A set is *finite* if it is either empty or equipotent to the set $\{0, 1, 2, \dots, n-1\}$ for a natural number n ; otherwise it is *infinite*. An infinite set that is equipotent to the set \mathbb{N} of all natural numbers is called *countably infinite*; otherwise the infinite set is *uncountable*. The term *countable* means either finite or countably infinite.

With the formal definition $0 = \emptyset$ and $n = \{0, 1, 2, \dots, n-1\}$ for $n \in \mathbb{N}$, a finite set is equivalently ‘equipotent to a whole number’. Each finite set X is equipotent to a *unique* whole number $|X| = n \in \mathbb{N}_0$, the ‘number of elements of X ’. In short, a finite set is a set consisting of a finite number of elements.

The property that each finite set is equipotent to a unique whole number may be extended to infinite sets. The generalized ‘number of elements’ of a set is called its *cardinality*, and formally one has the

0.10 Property Every set is equipotent to a unique *cardinal number*.

I will not go into the formal definition of cardinal number (and its related concept ordinal number) here. The interested reader may read Halmos [1960]. The usual partial order \leq of whole numbers may be extended to all cardinal numbers. One uses the same notation $|X| = n$ for the cardinality of the set X , where n may be an ‘infinite cardinal’ in addition to a whole number. Infinite cardinal numbers are usually denoted by the first letter \aleph (*aleph*) of the Hebrew alphabet. When $|X| = n$, one may simply say ‘ X has cardinal number n ’ or ‘ X has cardinality n ’.

When $|X| = n$, a bijective mapping from the cardinal number n (as a set) to the set X is called an *enumeration*, a ‘listing of the elements’ of X . While ‘to enumerate’ literally means ‘to count out’ (i.e. ‘to have a number as output’), the

domain of an enumeration may be any cardinal number, countable or uncountable. The enumeration map is not uniquely defined by the correspondence $n \leftrightarrow X$, since any permutation of the assignment also serves as a bijection (each different permutation—there being $n!$ of them for finite n —defining its own distinct listing of elements of the set X).

0.11 Theorem

- i. *Every set has a cardinal number.*
- ii. *Two sets A and B are equipotent if and only if they have the same cardinal number, that is, iff $|A| = |B|$.*
- iii. *$|A| \leq |B|$ if and only if A is equipotent to a subset of B .*
- iv. *$|A| < |B|$ if and only if A is equipotent to a subset of B but B is not equipotent to a subset of A .*

Some trivial properties of finite sets are:

0.12 Corollary

- i. *Every finite set has a unique number of elements.*
- ii. *Two finite sets are equipotent if and only if they have the same number of elements.*
- iii. *If a set is finite, then every one of its subsets is finite.*
- iv. *If a finite set X has n elements and a subset $A \subset X$ has k elements, then $k \leq n$; further, $k = n$ iff $A = X$.*
- v. *If a set is finite, then it is not equipotent to any of its proper subsets.*

Property v, that a finite set is not equipotent to any of its proper subsets, in fact characterizes finite sets. The inverse thus characterizes infinite sets; stated formally:

0.13 Theorem

- i. *A set is infinite if and only if it is equipotent to a proper subset of itself.*
- ii. *A set is finite if and only if it is not equipotent to any proper subset of itself.*

One also has the following concerning countability:

0.14 Lemma

- i. *Every subset of a countable set is countable.*
- ii. *Every infinite set has a countably infinite subset.*

0.15 Cardinality of the Power Set If $|X| = n$, then $|\mathcal{P}X| = 2^n$ (for all cardinal numbers n , finite and infinite). The proof is immediate from the fact that $\mathcal{P}X$ is

equipotent to $\mathbf{Set}(X, 2) = 2^X$, the hom-set of all mappings from X to $2 = \{0, 1\}$ (cf. *ML*: A.3). One may succinctly write

$$(5) \quad |2^X| = 2^{|X|}.$$

0.16 Cantor’s Continuum Hypothesis The cardinality of the set \mathbb{N} of all natural numbers (whence of all countably infinite sets) is denoted by \aleph_0 . In view of Lemma 0.14.ii, \aleph_0 is, then, the least infinite cardinal number. Analogously, the least uncountable cardinal number is usually denoted by \aleph_1 . In terms of the canonical order relation \leq of cardinal numbers, \aleph_1 is the *least* cardinal number strictly following \aleph_0 .

Cantor’s Theorem (Theorem 0.8) dictates that the set \mathbb{N} is equipotent to a subset of its power set $\mathcal{P}\mathbb{N}$, but is not equipotent to $\mathcal{P}\mathbb{N}$ itself. Whence, it follows from Theorem 0.11.iv and Section 0.15 that $\aleph_0 < |\mathcal{P}\mathbb{N}| = 2^{\aleph_0}$. Since \aleph_1 is the least cardinal number larger than \aleph_0 , one must have

$$(6) \quad \aleph_1 \leq 2^{\aleph_0}.$$

In his famous *continuum hypothesis*, Cantor conjectured that $\aleph_1 = 2^{\aleph_0}$. (The word ‘continuum’ is used because 2^{\aleph_0} is also the cardinal number of the set \mathbb{R} of all real numbers, the ‘cardinality of the continuum’, usually denoted $|\mathbb{R}| = c$.)

The consistency of the continuum hypothesis with the usual axioms of set theory has been proven, that is, the equality $\aleph_1 = 2^{\aleph_0}$ is non-contradictory. It has likewise been proven that the continuum hypothesis is independent of the usual axioms of set theory; that is, the inequality $\aleph_1 < 2^{\aleph_0}$ is also non-contradictory.

Indexed Sets

0.17 Indexed Family Let I and X be sets. A *family of elements in X indexed by I* is a mapping $x : I \rightarrow X$. The domain I is called the *index set* (note the noun adjunct ‘index’), an element $i \in I$ is called an *index*, the range $x(I) \subset X$ is called an *indexed set* (note the past participle ‘indexed’), and the value $x(i)$ of the mapping x at an index i , written as x_i (whence the element-chasing form of the mapping x may be written as $x : i \mapsto x_i$), is a *term* (or more precisely ‘the i th term’) of the family. Such a mapping is often denoted

$$(7) \quad \{x_i\}_{i \in I},$$

and the mapping is also called an *indexed family* (in X). Note that notation (7) represents the indexed family, which is a mapping, whence $\{x_i\}_{i \in I} \in X^I$, while the notation

$$(8) \quad \{x_i : i \in I\}$$

represents the indexed set (i.e. the range of the indexed family, whence $\{x_i : i \in I\} = x(I) \subset X$). The notion of ‘the i th term’ only makes sense with respect to the indexed family (7) but not the indexed set (8). Occasionally one may simply use $\{x_i\}$ for (7) if the index set I is implicitly understood, but this is not good notation (although it is commonly accepted) because of the possible equivocation between the two different entities (7) and (8), essentially the identification of a mapping with its range.

One may also note that the mapping $x : I \rightarrow x(I)$ is surjective (which is simply the statement that a mapping maps *onto* its range), but an indexed family is not required to be injective. Explicitly, it may happen for $i, j \in I$ that $i \neq j$ but $x_i = x_j$; that is, there may be ‘duplicated terms’.

0.18 Indexed Family of Sets An *indexed family of sets* is an indexed family $A : I \rightarrow \mathcal{P}X$ (of elements in $\mathcal{P}X$), denoted

$$(9) \quad \{A_i\}_{i \in I},$$

where each $A_i \subset X$.

0.19 Indexed Partition An *indexed partition* of a set X is an indexed family of *nonempty* sets $A : I \rightarrow \mathcal{P}X$ for which the collection of subsets $\{A_i : i \in I\}$ forms a *partition* of X (cf. *ML*: 1.16), that is, for each $i \in I$, $A_i \neq \emptyset$, and

$$(10) \quad X = \bigcup_{i \in I} A_i$$

with

$$(11) \quad A_i \cap A_j = \emptyset \quad \text{for } i \neq j.$$

0.20 Axiom of Choice If $\{A_i\}_{i \in I}$ is an indexed family of nonempty sets indexed by a *nonempty* index set I , then there exists an indexed family $\{x_i\}_{i \in I}$ such that for each $i \in I$, $x_i \in A_i$.

Compare this with the equivalent statement from *ML*: 1.37: Given a nonempty family \mathfrak{A} of nonempty sets, there is a mapping f with domain \mathfrak{A} such that $f(A) \in A$ for all $A \in \mathfrak{A}$. The correspondence is $\mathfrak{A} = \{A_i : i \in I\}$. The mapping $f : \mathfrak{A} \rightarrow \bigcup_{A \in \mathfrak{A}} A$ (i.e. $f : \{A_i : i \in I\} \rightarrow \bigcup_{i \in I} A_i$) defined by

$$(12) \quad f(A_i) = x_i$$

is called a *choice mapping*. When the index set I is finite, choosing, for each $i \in I$, an x_i from a nonempty set A_i (i.e. defining a choice mapping f) is a simple procedure; not so when I is infinite. There is no prescription of how infinitely many choices are to be made, and that is why the existence of the choice mapping has to be postulated in an axiom. It is almost a convention in mathematics that one explicitly acknowledges when a consequence depends on the Axiom of Choice.

Sequences

0.21 Sequence An indexed family $\{x_i\}_{i \in I}$ in X with an index set $I = \{1, 2, \dots, n\}$ (for some natural number n) or $I = \mathbb{N}$ is called a *sequence* (*finite* or *infinite*, respectively) *in* X .

A finite sequence is often written as a list of its terms:

$$(13) \quad \{x_i\}_{i \in \{1, 2, \dots, n\}} = \{x_1, x_2, \dots, x_n\};$$

so also is an infinite sequence:

$$(14) \quad \{x_i\}_{i \in \mathbb{N}} = \{x_1, x_2, x_3, \dots\}.$$

Note that in the listing of the elements on the right-hand side of each of (13) and (14), the distinction between indexed family and indexed set is already somewhat blurred (again, this is not good notation but is commonly accepted). A caveat of the blurred listing notation is that in an indexed family, duplicated terms are kept, while in an indexed set, duplicated terms are (almost) always eliminated. Consider, for example, a finite sequence of two vectors v_1 and v_2 , with the two

vectors identical and nonzero, that is, $v_1 = v_2 \neq 0$. The *sequence* (i.e. indexed family) $\{v_1, v_2\}$ is linearly dependent, but the *set* $\{v_1, v_2\} = \{v_1\}$ is linearly independent, since it consists actually of just one nonzero vector.

0.22 Preorder The terms of finite and infinite sequences are well-ordered (*ML*: 3.39) by the natural order of integers of their index sets ($\{1, 2, \dots, n\}$ and \mathbb{N} respectively). Thus, one may *truncate*, for example, an infinite sequence $\{x_1, x_2, x_3, \dots\}$ *after m terms* (where $m \in \mathbb{N}$) to split off the infinite ‘tail end’ $\{x_{m+1}, x_{m+2}, x_{m+3}, \dots\}$ and obtain the finite sequence $\{x_1, x_2, \dots, x_m\}$. One may say, as another example, that a term x_i *precedes* another term x_j if $i < j$ (and that x_j *follows* x_i in the sequence).

It is important to note that the ordering of the terms in a sequence $\{x_i\}$ has to do with the *positions* of the terms, and not the ordering of the elements themselves in the indexed set. This is because the codomain X is not necessarily equipped with an order, and unless it is, a statement such as $x_i \leq x_j$ is meaningless.

There is, however, a way to *define* the binary relation of *precedence* on the range $x(I) = \{x_i : i \in I\}$ using the order inherent in $I \subset \mathbb{N}$, by

$$(15) \quad x_i \preceq x_j \text{ in } x(I) \text{ iff } i \leq j \text{ in } \mathbb{N}.$$

Note that the relation \preceq is defined only on the range $x(I)$ and not on the rest of the codomain $X \sim x(I)$. It is easy to see that the relation of precedence on $x(I)$ is reflexive and transitive (*ML*: 1.10), but not necessarily either antisymmetric or symmetric. A relation that is reflexive and transitive is called a *preorder* (something that is ‘not quite’ a partial order, *ML*: 1.20, or an equivalence, *ML*: 1.11). A set equipped with a preorder is called a *preordered set* or *proset*. (I shall revisit binary relations, especially these with special properties, in Chapter 3 of this book.)

Each preordered set $\langle S, \preceq \rangle$ is itself a category (*cf.* *ML*: A.1). This category S has objects the elements of S , and for $a, b \in S$, the hom-set $S(a, b)$ either contains a single S -morphism or is empty, depending on whether $a \preceq b$ or not. Transitivity of \preceq provides for the composition of morphisms, and reflexivity provides the identity morphisms in $S(a, a)$. (I have discussed *poset*, i.e. partially ordered set, as category in *ML*: 1.31, but indeed a *proset* suffices; the antisymmetry is not needed.)

0.23 Monotonic Sequence As a mapping $x: \langle I, \leq \rangle \rightarrow \langle x(I), \preceq \rangle$ of posets, the sequence $\{x_i\}$ preserves the ordering relation by the very definition of \preceq in (15), and is therefore a morphism in the category of posets and order-preserving mappings. But the order-preserving property of a sequence $\{x_i\}$ may also exist, as a mapping $x: \langle I, \leq \rangle \rightarrow \langle x(I), \preceq \rangle$ of posets, when the codomain X is already equipped with its own preorder \preceq (even when \preceq is not the precedence defined on $x(I) \subset X$; in particular, when $\langle X, \preceq \rangle$ is in fact a *poset*). As the mapping $x: \langle I, \leq \rangle \rightarrow \langle X, \preceq \rangle$ of posets, the sequence $\{x_i\}$ is *isotone* (cf. *ML*: 1.23) if

$$(16) \quad i \leq j \text{ in } I \Rightarrow x_i \preceq x_j \text{ in } X.$$

An isotone sequence $\{x_i\}$ is more commonly called *monotonically increasing*, and implication (16) is equivalent to

$$(17) \quad x_i \preceq x_{i+1} \text{ for } i \in \mathbb{N}$$

(or for $i \in \{1, 2, \dots, n-1\}$ in the case of a finite sequence). (The sequence $\{x_i\}$ is, of course, monotonically increasing with respect to the relation of precedence on $x(I)$.) If the mapping $x: \langle I, \leq \rangle \rightarrow \langle X, \preceq \rangle$ is order reversing (i.e. ‘antitone’), then

$$(18) \quad i \leq j \text{ in } I \Rightarrow x_i \succeq x_j \text{ in } X,$$

which is equivalent to

$$(19) \quad x_i \succeq x_{i+1} \text{ for } i \in \mathbb{N}$$

(or, again, for $i \in \{1, 2, \dots, n-1\}$ in the case of a finite sequence); such is a *monotonically decreasing* sequence.

If the ordering in (17) is strict, that is, $x_i < x_{i+1}$ which means ‘ $x_i \preceq x_{i+1}$ and $x_i \neq x_{i+1}$ ’ (*ML*: 1.22), then the sequence $\{x_i\}$ is *strictly increasing*. Likewise, a strict inequality $x_i > x_{i+1}$ ($x_i \succeq x_{i+1}$ and $x_i \neq x_{i+1}$) in (19) defines a *strictly decreasing* sequence. The class of *monotonic sequences* consists of all the increasing and the decreasing sequences.

0.24 Subsequence There is an important way of obtaining new sequences from a given infinite sequence $\{x_i\}_{i \in \mathbb{N}} = \{x_1, x_2, x_3, \dots\}$. Let $\{n_k\}_{k \in \mathbb{N}} = \{n_1, n_2, n_3, \dots\}$ be an infinite sequence in \mathbb{N} such that

$$(20) \quad n_{k+1} > n_k \quad \text{for } k \in \mathbb{N}$$

(i.e. the sequence $n: \langle \mathbb{N}, \leq \rangle \rightarrow \langle \mathbb{N}, \leq \rangle$ is strictly increasing). The sequence $k \mapsto x_{n_k}$ is called a *subsequence* of the sequence $\{x_i\}_{i \in \mathbb{N}} = \{x_1, x_2, x_3, \dots\}$ and is denoted

$$(21) \quad \{x_{n_k}\}_{k \in \mathbb{N}} = \{x_{n_1}, x_{n_2}, x_{n_3}, \dots\}.$$

One may see that the subsequence $\{x_{n_k}\}_{k \in \mathbb{N}}$ is simply the (sequential) composition (*ML*: 5.13) of the mapping $k \mapsto n_k$ (in $\mathbb{N}^{\mathbb{N}}$) followed by the mapping $n \mapsto x_n$ (in $X^{\mathbb{N}}$).

One may also readily verify that every sequence $\{x_i\}$ is a subsequence of itself, and if $\{z_i\}$ is a subsequence of $\{y_i\}$ and $\{y_i\}$ is a subsequence of $\{x_i\}$, then $\{z_i\}$ is a subsequence of $\{x_i\}$. Stated otherwise, the relation ‘is a subsequence of’ on the set $X^{\mathbb{N}}$ of all infinite sequences in X is reflexive and transitive; it is, therefore, a preorder (Section 0.22). Trivially, the relation ‘is a subsequence of’ is not symmetric, so it is not an equivalence relation; that it is not antisymmetric (whence not a partial order) may be seen in the following example. Let

$$(22) \quad \{x_i\} = \{0, 1, 0, 1, 0, 1, \dots\} \quad \text{and} \quad \{y_i\} = \{1, 0, 1, 0, 1, 0, \dots\}.$$

The mapping $n: k \mapsto k+1$, that is, the sequence

$$(23) \quad \{n_k\} = \{2, 3, 4, 5, 6, 7, \dots\},$$

is such that

$$(24) \quad y \circ n = x \quad \text{and} \quad x \circ n = y,$$

that is,

$$(25) \quad \{y_{n_k}\} = \{x_i\} = \{0, 1, 0, 1, 0, 1, \dots\} \quad \text{and} \quad \{x_{n_k}\} = \{y_i\} = \{1, 0, 1, 0, 1, 0, \dots\}.$$

So $\{x_i\}$ and $\{y_i\}$ are subsequences of each other, but $\{x_i\} \neq \{y_i\}$.

0.25 Enumerating Sequence Recall (Definition 0.9) that a nonempty set is finite if it is equipotent to the set $\{0, 1, 2, \dots, n-1\} \cong \{1, 2, \dots, n\}$ for a natural number n , and a set is countably infinite if it is equipotent to the set \mathbb{N} of all natural numbers. Equipotence implies that a nonempty finite set X with cardinality $|X| = n$ has a *bijective indexed family* $x: \{1, 2, \dots, n\} \rightarrow X$ listing its elements in order and representing it as a finite sequence $\{x_i\}_{i \in \{1, 2, \dots, n\}} = \{x_1, x_2, \dots, x_n\}$. (This means, in particular, that for $i, j \in I$, if $i \neq j$, then $x_i \neq x_j$.) Similarly, a countably infinite set X (with cardinality $|X| = \aleph_0$) may be represented as an infinite sequence $\{x_i\}_{i \in \mathbb{N}} = \{x_1, x_2, x_3, \dots\}$ with its corresponding *bijective indexed family* $x: \mathbb{N} \rightarrow X$. The bijective indexed family x , a mapping turning a countable set into a sequence, is called an *enumeration* of X (*cf.* Section 0.10).

For both finite and infinite sets, the choice of the enumeration is, as previously mentioned, not unique: any permutation of the assignment also serves as an enumeration (each different permutation defining its own distinct listing of elements and sequential representation of the set).