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Stochastic Differential Inclusions and Applications

Springer

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To my wife with love
and gratitude for support

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Preface

There has been a great deal of interest in optimal control systems described by stochastic and partial differential equations. These optimal control problems lead to stochastic and partial differential inclusions. The aim of this book is to present a unified theory of stochastic differential inclusions written in integral form with both types of stochastic set-valued integrals defined as subsets of the space $\mathbb{L}^2(\Omega, \mathbb{R}^n)$ and as multifunctions with closed values in the space $\mathbb{R}^n$. Such defined inclusions are therefore divided into two types: stochastic functional inclusions ($SFI(F,G)$) and stochastic differential inclusions ($SDI(F,G)$), respectively. The main results of the book deal with properties of solution sets of stochastic functional inclusions and some of their applications in stochastic optimal control theory and in the theory of partial differential inclusions. In particular, apart from the existence of weak solutions for initial value problems of stochastic functional inclusions, the existence of their strong and weak viable solutions is also investigated. An important role in applications is played by theorems on weak compactness of solution sets of weak and viable weak solutions for the above initial value problems. As a result of these properties, some optimal control problems for dynamical systems described by stochastic and partial differential inclusions are obtained. Let us remark that for a given pair (F,G) of multifunctions, the sets $\mathcal{X}(F,G)$ and $\mathcal{S}(F,G)$ of all weak solutions of $SFI(F,G)$ and $SDI(F,G)$, respectively, are defined as families of systems $(\mathcal{P}_{\mathbb{F}}, x, B)$ consisting of a filtered probability space $\mathcal{P}_{\mathbb{F}}$, a continuous process $x = (x_t)_{t\geq 0}$, and an $\mathbb{F}$-Brownian motion $B = (B_t)_{t\geq}$ satisfying these inclusions. Immediately from the definitions of $SFI(F,G)$ and $SDI(F,G)$, it follows that $\mathcal{X}(F,G) \subset \mathcal{S}(F,G)$. It is natural to extend the results of this book to the set $\mathcal{S}(F,G)$ and consider weak solutions with x a càdlàg process instead of a continuous one. These problems are quite complicated and need new methods. Therefore, in this book, they are left as open problems.

The first papers dealing with stochastic functional inclusions written in integral form are due to Hiai [38] and Kisielewicz [50–56,58,60–62]. Independently, Ahmed [2], Da Prato and Frankowska [23], Aubin and Da Prato [9], and Aubin et al. [10] have considered stochastic differential inclusions symbolically written in the differential form $dx_t \in F(t,x_t)dt + G(t,x_t)dB_t$ and understood as a problem consisting in finding a system $(\mathcal{P}_{\mathbb{F}}, x, B)$ consisting of a filtered probability space $\mathcal{P}_{\mathbb{F}}$, a continuous process $x = (x_t)_{t \geq 0}$, and an $\mathbb{F}$-Brownian motion such that $x_t = x_0 + \int_0^t f_\tau \mathrm{d}\tau + \int_0^t g_\tau \mathrm{d}B_\tau$ with $f_t \in (F \circ x)_t =: F(t, x_t)$ and $g_t \in (G \circ x)_t =: F(t, x_t)$ a.s. for $t \geq 0$. Stochastic functional inclusions defined by Hiai [38] and Kisielewicz [51] are in the general case understood as a problem consisting in finding a system $(\mathcal{P}_{\mathbb{F}}, x, B)$ such that $x_t - x_s \in \mathrm{cl}_{\mathbb{L}}\{J_{st}(F \circ x) + \mathcal{J}_{st}(G \circ x)\}$ for every $0 \leq s \leq t < \infty$, where $J_{st}(F \circ x)$ and $\mathcal{J}_{st}(G \circ x)$ denote set-valued functional integrals on the interval $[s, t]$ of $F \circ x$ and $G \circ x$, respectively. It is evident that some properties of stochastic functional inclusions written in integral form follow from properties of set-valued stochastic integrals. Such properties are difficult to obtain for stochastic differential inclusions written in differential form.

The first results dealing with set-valued stochastic integrals with respect to the Wiener process with application to some set-valued stochastic differential equations are due to Bocşan [22]. More general definitions and properties of set-valued stochastic integrals were given in the above-cited papers of Hiai and Kisielewicz, where set-valued stochastic integrals are defined as certain subsets of the spaces $\mathbb{L}^2(\Omega, \mathbb{R}^n)$ and $\mathbb{L}^2(\Omega, \mathcal{X})$ of all square integrable random variables with values at $\mathbb{R}^n$ and $\mathcal{X}$, respectively, where $\mathcal{X}$ is a Hilbert space. In this book, such integrals are called stochastic functional set-valued integrals. Unfortunately, such integrals do not admit a representation by set-valued random variables with values in $\mathbb{R}^n$ and $\mathcal{X}$, because they are not decomposable subsets of $\mathbb{L}^2(\Omega, \mathbb{R}^n)$ and $\mathbb{L}^2(\Omega, \mathcal{X})$, respectively. Later, Jung and Kim [46] (see also [98]) defined a set-valued stochastic integral as a set-valued random variable determined by a closed decomposable hull of the above-mentioned set-valued stochastic functional integral. Unfortunately, the authors did not obtain any properties of such integrals. In Chap. 3, we apply the above approach to the theory of set-valued stochastic integrals of $\mathbb{F}$-nonanticipative multiprocesses and obtain some properties of such integrals.

The first results dealing with partial differential inclusions were in fact simple generalizations of ordinary differential inclusions. They dealt with hyperbolic partial differential inclusions of the form $z''_{x,y} \in F(x, y, z)$. Later on, partial differential inclusions $z''_{x,y} \in F(x, y, z, z'_x, z'_y)$ were also investigated. Such partial differential inclusions have been considered by Kubiaczyk [65], Dawidowski and Kubiaczyk [24], Dawidowski et al. [25], and Sosulski [92,93], among others. Some hyperbolic partial differential inclusions were considered in Aubin and Frankowska [11]. A new idea dealing with partial differential inclusions was given by Bartuzel and Fryszkowski in their papers [15–17], where partial differential inclusions of the form $Du \in F(u)$ with a lower semicontinuous multifunction F and a partial differential operator D are considered. The existence and properties of solutions of initial and boundary value problems of such inclusions follow from classical results dealing with abstract differential inclusions. As usual, certain types of continuous selection theorems for set-valued mappings play an important role in investigations of such inclusions.

The partial differential inclusions considered in this book have the forms $u'_t(t, x) \in (\mathbb{L}_{FG}u)(t, x) + c(t, x)u(t, x)$ and $\psi(t, x) \in (\mathbb{L}_{FG}u)(t, x) + c(t, x)u(t, x)$, where c and ψ are given functions and $\mathbb{L}_{FG}$ denotes the set-valued diffusion generator defined by given multifunctions F and G. The first results dealing with such partial differential inclusions are due to Kisielewicz [60, 61]. The initial and boundary value problems of such inclusions are investigated by stochastic methods. Their solutions are characterized by weak solutions of stochastic functional inclusions $SFI(F, G)$. Such an approach leads to natural methods of solving some optimal control problems for systems described by the above type of partial differential inclusions. It is a consequence of weak compactness with respect to the convergence in distribution of sets of all weak solutions of considered stochastic functional inclusions.

The content of the book is divided into seven parts. Chapter 1 covers basic notions and theorems of the theory of stochastic processes. Chapter 2 contains the fundamental notions of the theory of set-valued mappings and the theory of set-valued stochastic processes. Chapter 3 is devoted to the theory of set-valued stochastic integrals. Apart from their properties, it contains some important selection theorems. The main results of Chap. 4 deal with properties of stochastic functional and differential inclusions. In particular, it contains theorems dealing with weak compactness with respect to convergence in distribution of solution sets of weak solutions of initial value problems for stochastic functional inclusions. Chapter 5 contains some results dealing with viability theory for forward and backward stochastic functional and differential inclusions, whereas Chaps. 6 and 7 are devoted to some applications of the above-mentioned results to partial differential inclusions and to some optimal control problems for systems described by stochastic functional and partial differential inclusions.

The present book is intended for students, professionals in mathematics, and those interested in applications of the theory. Selected probabilistic methods and the theory of set-valued mappings are needed for understanding the text. Formulas, theorems, lemmas, remarks, and corollaries are numbered separately in each chapter and denoted by pairs of numbers. The first stands for the section number, the second for the number of the formula, theorem, etc. If we need to quote some formula or theorem given in the same chapter, we always write only this pair. In other cases, we will use this pair with information indicated the chapter number. The ends of proofs, theorems, remarks, and corollaries are denoted by $\square$.

The manuscript of this book was read by my colleagues M. Michta and J. Motyl, who made many valuable comments. The last version of the manuscript was read by Professor Diethard Pallaschke. His remarks and propositions were very useful in my last correction of the manuscript. It is my pleasure to thank all of them for their efforts.

Zielona Góra, Poland Michał Kisielewicz

List of Symbols

$\mathbb{F}$	– filtration of a probability space $(\Omega, \mathcal{F}, P)$, 1
$\mathcal{P}_{\mathbb{F}}$	– filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, 1
$\mathbb{R}^{+}$	– set of all non-negative real numbers, 2
$\in$	– is an element of, 1
$\mathbb{R}^{n}$	– n-dimensional Euclidean spaces, 2
$C(\mathbb{R}^{+}, \mathbb{R}^{n})$	– metric space of continuous functions, 2
$\mathcal{D}(\mathbb{R}^{+}, \mathbb{R}^{n})$	– metric space of càdlàg functions, 2
Q	– set of all rational numbers, 2
$\subset$	– subset of (set inclusion relation), 3
$\cap$	– intersection of sets, 2
$\cup$	– union of sets, 2
$A \setminus B$	– complement of B with respect to A, 3
$\notin$	– is not an element of, 3
τ_D^X	– first exit time of a stochastic process X from a set D, 3
$S \wedge T$	– minimum of stopping times S and T, 3
$S \vee T$	– maximum of stopping times S and T, 3
$\mathcal{F}_T$	– σ-algebra induced by a stopping time T, 3
cad($\mathbb{F}$)	– family of $\mathbb{F}$-adapted càdlàg processes, 3
$\sigma(\mathcal{M})$	– σ-algebra generated by a family $\mathcal{M}$ of random variables, 4
$\beta(\mathcal{X})$	– Borel σ-algebra of subsets of a metric space $(\mathcal{X}, \rho)$, 4
$\mathcal{M}(\mathcal{X})$	– space of probability measures on $\beta(\mathcal{X})$, 4
$P_n \Rightarrow P$	– weak convergence of a sequence of probability measures, 4
PX^{-1}	– distribution of a random variable X, 6
$X_n \xrightarrow{P} X$	– convergence in probability of a sequence of random variables, 6
$X_n \to X \; a.s.$	– convergence a.s. of a sequence of random variables, 6
$X_n \Rightarrow X$	– convergence in distribution of a sequence of random variables, 6

$\beta(\mathbb{R}^+)\otimes\mathcal{F}$	– product σ-algebra of σ-algebras $\beta(\mathbb{R}^+)$ and $\mathcal{F}$, 11
$\mathcal{P}(\mathbb{F})$	– $\mathbb{F}$-predictable σ-algebra, 11
$\mathcal{O}(\mathbb{F})$	– $\mathbb{F}$-optional σ-algebra, 11
$\mathcal{G}(C)$	– σ-algebra of cylindrical sets of $C(\mathbb{R}^+,\mathbb{R}^n)$, 12
$\mathcal{G}(\mathcal{D})$	– σ-algebra of cylindrical sets of $\mathcal{D}(\mathbb{R}^+,\mathbb{R}^n)$, 12
$C_+(\mathbb{R}^+,\mathbb{R}^n)$	– metric space of right continuous functions $x:\mathbb{R}^+\to\mathbb{R}^n$, 12
$C_-(\mathbb{R}^+,\mathbb{R}^n)$	– metric space of left continuous functions $x:\mathbb{R}^+\to\mathbb{R}^n$, 12
$E[Y\|\mathcal{F}_t]$	– conditional expectation of a random variable Y, 22
$\mathbb{I}_{\{T<t\}}$	– characteristic function of a random set $\{T<t\}$, 22
X^T	– process stopped at T, 22
$\langle X,Y\rangle$	– cross-variation of X and Y, 25
$\langle X\rangle$	– quadratic variation of X, 25
$\|\Delta\|$	– diameter of a partition of the interval $[0,T]$, 25
$\mathbb{N}$	– set of all nonnegative integers, 27
$(N_t)_{t\ge 0}$	– Poisson process, 27
$(B_t)_{t\ge 0}$	– Brownian motion, 28
$\mathcal{M}^2_{\mathbb{F}}(a,b)$	– space of some $\mathbb{F}$-nonanticipative processes, 32
$\mathcal{L}^2_{\mathbb{F}}(a,b)$	– space of some $\mathbb{F}$-nonanticipative processes, 32
$\mathcal{S}_{\mathbb{F}}(a,b)$	– space of simple processes of $\mathcal{M}^2_{\mathbb{F}}(a,b)$, 32
$\mathbb{L}^p(\Omega,\mathbb{R}^n)$	– space $\mathbb{L}^p(\Omega,\mathcal{F},P,\mathbb{R}^n)$, 35
dX	– stochastic differential of an Itô process $X=(X_t)_{t\ge 0}$, 40
$\mathbb{R}^{d\times m}$	– space of $\mathrm{d}\times\mathrm{m}$-matrices, 43
$A\Delta B$	– symmetric difference of A and B, 42
$\mathbb{L}_{fg}$	– semi-elliptic partial differential operator, 44
$(\varphi^h_t)_{t\ge 0}$	– continuous local martingale on $\mathcal{P}_{\mathbb{F}}$, 44
Q^x	– probability law of Itô diffusion starting with $(0,\mathrm{x})$, 51
$Q^{s,x}$	– probability law of Itô diffusion starting with (s,x), 51
E^x	– mean value operator with respect to Q^x, 51
$E^{s,x}$	– mean value operator with respect to $Q^{s,x}$, 51
$\mathcal{A}_X$	– infinitesimal generator of an Itô diffusion X, 54
$\mathcal{L}_X$	– characteristic operator of an Itô diffusion X, 54
τ_H	– first exit time of an Itô diffusion from a set H, 56
$\mathrm{Lim\,inf}\,A_n$	– limit inferior of a sequence $(A_n)_{n=1}^\infty$ of sets, 67
$\mathrm{Lim\,sup}\,A_n$	– limit superior of a sequence $(A_n)_{n=1}^\infty$ of sets, 67
$\mathrm{Li}\,A_n$	– Kuratowski limit inferior of a sequence $(A_n)_{n=1}^\infty$ of sets, 67
$\mathrm{Ls}\,A_n$	– Kuratowski limit superior of a sequence $(A_n)_{n=1}^\infty$ of sets, 68
$\mathrm{Cl}(X)$	– space of all nonempty closed subsets of a metric space X, 68
$\mathrm{h}(A,B)$	– Hausdorff distance of $A,B\in\mathrm{Cl}(X)$, 68
$\mathrm{dist}(a,A)$	– distance of a point $a\in X$ to a set A, 69
$\mathcal{P}(X)$	– space of all nonempty subsets of a metric space X, 70

l.s.c. – lower semicontinuity, 71
u.s.c. – upper semicontinuity, 71
H − l.s.c. – lower semicontinuity with respect to the Hausdorff metric, 71
H − u.s.c. – upper semicontinuity with respect to the Hausdorff metric, 70
Comp(Y) – space of all nonempty compact subsets of a topological space Y, 71
$\sigma(\cdot, A)$ – support function of a set $A \subset \mathbb{R}^d$, 77
Conv($\mathbb{R}^d$) – space of all nonempty compact convex subsets of $\mathbb{R}^d$, 77
s(A) – Steiner point of a set $A \in \mathrm{Conv}(\mathbb{R}^d)$, 78
$\langle \cdot, \cdot \rangle$ – inner product in the space $\mathbb{R}^d$, 78
Graph(F) – graph of a multifunction F, 82
cl(A) – closure of a subset A of a topologigal space, 83
$S(F)$ – set of all selectors $f \in \mathbb{L}^p(T, \mathbb{R}^d)$ of a multifunction F, 84
$\mathcal{M}(T, \mathbb{R}^d)$ – space of all measurable multifunctions $F : T \to \mathrm{Cl}(\mathbb{R}^d)$, 84
$\mathcal{A}(T, \mathbb{R}^d)$ – subset of $\mathcal{M}(T, \mathbb{R}^d)$ such that $S(F) \neq \emptyset$, 84
$\overline{\mathrm{co}}\, S(F)$ – closed convex hull of $S(F)$, 85
dec$\{C\}$ – decomposable hull of a set $C \subset \mathbb{L}^p(T, \mathbb{R}^d)$, 89
$\overline{\mathrm{dec}}\{C\}$ – closed decomposable hull of a set $C \subset \mathbb{L}^p(T, \mathbb{R}^d)$, 89
$\Sigma_{\mathbb{F}}$ – σ-algebra of $\mathbb{F}$-nonanticipative subsets of $T \times \Omega$, 96
$S_{\mathbb{F}}(\Phi)$ – set of all $\mathbb{F}$-nonaticipatine selectors of multifunction Φ, 96
$\mathcal{M}(\mathbb{R}^+ \times \Omega, \mathbb{R}^d)$ – space of all measurable set-valued processes, 97
$\mathcal{M}_{\mathbb{F}}(\mathbb{R}^+ \times \Omega, \mathbb{R}^d)$ – space of $\Sigma_{\mathbb{F}}$-measurable multifunctions, 97
$\mathcal{L}^2(\mathbb{R}^+ \times \Omega, \mathbb{R}^d)$ – space of measurable square integrable multifunctions, 97
$\mathcal{L}^2_{\mathbb{F}}(\mathbb{R}^+ \times \Omega, \mathbb{R}^d)$ – space of square integrable $\Sigma_{\mathbb{F}}$-measurable multifunctions, 97
$E[\Phi|\mathcal{G}]$ – $\mathcal{G}$-conditional expectation of a set-valued mapping Φ, 99
J – linear mapping defined by $J(\phi) = (\int_0^T \phi_t \mathrm{d}t)(\cdot)$, 103
$\mathcal{J}$ – linear mapping defined by $\mathcal{J}(\psi) = (\int_0^T \psi_t \, \mathrm{d}B_t)(\cdot)$, 103
$(\mathcal{A}) \int_0^T \Phi_t \mathrm{d}t$ – set-valued stochastic Aumann′s integral, 115
$\int_0^T \Phi_t dt$ – set-valued stochastic Aumann′s integral, 115
$\int_0^T \Psi_t dB_t$ – set-valued stochastic Itô integral, 115
$D(\Psi)$ – set-valued mapping $D(\Psi)_t(\omega) = \{v \cdot v^* : v \in \Psi_t(\omega)\}$, 133
$\mathcal{C}_r$ – metric space of continuous functions $\varphi : \mathbb{R}^r \to \mathbb{R}^r$, 133
$\mathcal{C}_{r\times r}$ – metric space of continuous functions $\psi : \mathbb{R}^r \to \mathbb{R}^{r\times r}$, 133
$\varphi(h)$ – gradient of a function $h \in C_0^2(\mathbb{R}^r, \mathbb{R})$, 133
$\psi(h)$ – matrix of second partial derivatives of $h \in C_0^2(\mathbb{R}^r, \mathbb{R})$, 133

$\mathbb{L}^x_{fg}(\varphi,\psi)$	–	semi-elliptic differential operator, 136
$\mathbb{L}^x_{fg}h$	–	semi-elliptic differential operator, 136
$SFI(F,G)$	–	stochastic functional inclusion, 147
$\overline{SFI}(F,G)$	–	stochastic functional inclusion, 147
$\mathcal{X}_\mu(F,G)$	–	set of all weak solutions of $SFI(F,G)$, 148
$\overline{\mathcal{X}}_\mu(F,G)$	–	set of all weak solutions of $\overline{SFI}(F,G)$, 148
$\mathbb{L}^x_{fg}$	–	semi-elliptic partial differential operator, 151
$\mathbb{L}^x_{AB}$	–	set of all $\mathbb{L}^x_{fg}$ for $(f,g)\in A\times B$, 151
$\mathcal{M}^x_{AB}$	–	family of all $\mathbb{L}^x_{fg}\in\mathbb{L}^x_{AB}$ generating local martingales, 152
$\mathbb{L}^x_{FG}$	–	set $\mathbb{L}^x_{S_{\mathbb{F}}(F\circ x)S_{\mathbb{F}}(G\circ x)}$, 152
$SDI(F,G)$	–	stochastic differential inclusion, 163
$\overline{SDI}(F,G)$	–	stochastic differential inclusion, 163
$BSDI(F,H)$	–	backward stochastic differential inclusion, 165
$\mathcal{B}(F,H)$	–	set of all weak solutions of $BSDI(F,H)$, 166
$\mathcal{CB}(F,H)$	–	set of all continuous weak solutions of $BSDI(F,H)$, 166
$\mathcal{S}(\mathbb{F},\mathbb{R}^d)$	–	space of d-dimensional continuous $\mathbb{F}$-semimartingales, 167
$\mathbb{D}(\mathbb{F},\mathbb{R}^d)$	–	space of d-dimensional $\mathbb{F}$-adapted càdlàg processes, 167
$\mathbf{C}(\mathbb{F},\mathbb{R}^{\mathbf{d}})$	–	space of d-dimensional $\mathbb{F}$-adapted continuous processes, 167
$\mathcal{T}_K(t,x)$	–	stochastic tangent set to $K\subset\mathbb{R}^d$, 198
$\mathcal{S}_K(t,x)$	–	stochastic tangent set to $K\subset\mathbb{R}^d$, 199
$\tau_K(t,x)$	–	stochastic contingent set to $K\subset\mathbb{R}^d$, 200
$\mathcal{A}_{fg}$	–	infinitesimal diffusion generator, 217
$\mathbb{L}_{FG}$	–	set-valued semi-elliptic partial differential operator, 218
$\mathcal{C}(F),\mathcal{C}(G)$	–	sets of continuous selectors of multifunctions F and G, 218
$\mathcal{A}_{FG}$	–	set of the form $\{\mathcal{A}_{fg}:(f,g)\in\mathcal{C}(F)\times\mathcal{C}(G)\}$, 218
$\mathcal{L}_{FG}$	–	set of the form $\{\mathcal{L}_{fg}:(f,g)\in\mathcal{C}(F)\times\mathcal{C}(G)\}$, 218