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The Sherrington-Kirkpatrick Model

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Preface

This book is an attempt to give, as much as possible, a self-contained presentation of some of the main ideas involved in the mathematical analysis of the Sherrington–Kirkpatrick model and closely related mixed p-spin models of spin glasses. Certain topics, such as the high-temperature region and phase transition, are not covered and can be found in the comprehensive manuscript of Michel Talagrand [66].

In 1975 David Sherrington and Scott Kirkpatrick introduced in [58] a model of a spin glass—a disordered magnetic alloy that exhibits unusual magnetic behavior. This model is also often interpreted as a question about a typical behavior of the optimization problem $\max_{\sigma\in\Sigma_N} H_N(\sigma)$ for a certain function $H_N(\sigma)$ on the space of *spin configurations* $\Sigma_N = \{-1,+1\}^N$. This means that the parameters of $H_N(\sigma)$ are modeled as random variables and one would like to understand the asymptotic behavior of the average $\mathbb{E}\max_{\sigma\in\Sigma_N} H_N(\sigma)$ in the *thermodynamic (infinite-volume) limit*, as the size of the system N goes to infinity. We will see in Chap. 1 that, in order to solve this problem, it is enough to compute the limit of the *free energy*,

$$\lim_{N\to\infty} \frac{1}{N}\mathbb{E}\log \sum_{\sigma\in\Sigma_N} \exp \beta H_N(\sigma),$$

for each *inverse temperature* parameter $\beta = 1/T > 0$, and the formula for this limit was proposed by Sherrington and Kirkpatrick in [58] based on the so-called replica formalism. At the same time, they observed that their *replica symmetric solution* exhibits "unphysical behavior" at low temperature, which means that it can only be correct at high temperature.

Several years later Giorgio Parisi proposed [51, 52] another *replica symmetry breaking solution* within replica theory, now called the *Parisi ansatz*, which was consistent at any temperature $T \geq 0$ and, moreover, was in excellent agreement with computer simulations. The key feature of the celebrated Parisi ansatz was the choice of an *ultrametric parametrization* of the replica matrix in the computation of the free energy based on the replica approach. A rich theory emerged in the physics literature during the subsequent interpretation of the Parisi solution in terms of some physical properties of the *Gibbs measure* of the model

$$G_N(\sigma) = \frac{\exp \beta H_N(\sigma)}{\sum_{\rho \in \Sigma_N} \exp \beta H_N(\rho)}.$$

In particular, in the work of Parisi [53], the *order parameter* in the ultrametric parametrization of the replica matrix was related to the distribution of the *overlap* $R_{1,2} = N^{-1} \sum_{i=1}^N \sigma_i^1 \sigma_i^2$ of two spin configurations $\sigma^1, \sigma^2 \in \Sigma_N$ sampled from the Gibbs measure. The Parisi ansatz was further interpreted in terms of the geometric structure of the Gibbs measure in the work of Mézard et al. [37, 38], where it was understood, for example, that the ultrametricity of the replica matrix corresponds to the ultrametricity of the support of the Gibbs measure in the infinite-volume limit. Such reinterpretation of the Parisi solution formed a beautiful general physical theory of the model, which was described in the famous book of Mézard, Parisi, and Virasoro, "Spin Glass Theory and Beyond," [40]. In some sense, this also opened a path to a rigorous mathematical theory of the model.

Around the same time, motivated by the developments in the SK model, Bernard Derrida proposed two simplified models of spin glasses—the random energy model, REM, in [16, 17], and the generalized random energy model, GREM, in [18, 19]. The REM can be viewed as a formal limit of the family of the so-called pure p-spin models, in which the SK model corresponds to $p = 2$, and its Hamiltonian $H_N(\sigma)$ is given by an i.i.d. sequence of Gaussian random variables with variance N indexed by $\sigma \in \Sigma_N$, which is a rather classical object. The GREM combines several random energy models in a hierarchical way with the ultrametric structure built into the model from the beginning. Even though these simplified models do not shed light on the Parisi ansatz in the SK model directly, the structure of the Gibbs measures in these models was predicted to be, in some sense, identical to that of the SK model in the infinite-volume limit. For example, Derrida and Toulouse showed in [20] that the Gibbs weights in the REM have the same distribution in the thermodynamic limit as the Gibbs weights of the pure states (clusters of spin configurations) in the SK model; this latter distribution was computed earlier in [37] using the replica method. Independently, Mézard et al. [39] illustrated the connection between the REM and the SK model from a different point of view and, finally, de Dominicis and Hilhorst [15] demonstrated a similar connection between the Gibbs measure of the GREM and the global structure of the Gibbs measure in the SK model predicted by the Parisi ansatz.

The realization that the structure of the Gibbs measure in the SK model predicted by the Parisi replica theory coincides with the structure of the Gibbs measure in the GREM, which is much simpler than the SK model, turned out to be a very important step toward a deeper understanding of the Parisi ansatz. In particular, motivated by this connection with the SK model, in his seminal paper [56], David Ruelle gave an alternative explicit description of the Gibbs measure in the GREM in the thermodynamic limit in terms of a certain family of Poisson processes. As a result, one could now study the properties of these measures, nowadays called the *Ruelle probability cascades*, using the entire arsenal of the theory of Poisson processes. Some of these properties were already described in the original paper of Ruelle [56], while other important properties, which express certain invariance features of these measures, were discovered later by Erwin Bolthausen and Alain-Sol Sznitman in [10]. We will study the Ruelle probability cascades, including their invariance properties, in Chap. 2.

Another breakthrough in the mathematical analysis of the SK model came at the end of the nineties with the discovery of the two so-called *stability properties* of the Gibbs measure in the SK model in the work of Stefano Ghirlanda and Francesco Guerra [25] and Michael Aizenman and Pierluigi Contucci [1]. It was clear that these stability properties, known as the Ghirlanda–Guerra identities and the Aizenman–Contucci stochastic stability, impose very strong constraints on the structure of the Gibbs measure, but the question was whether they lead all the way to the Ruelle probability cascades. The Aizenman–Contucci stochastic stability is identical to one part of the Bolthausen–Sznitman invariance property for the Ruelle probability cascades. The fact that the Ghirlanda–Guerra identities also hold for the Ruelle probability cascades was first proved by Michel Talagrand in [62] in the case corresponding to the REM and, soon after, by Anton Bovier and Irina Kurkova [11] in the general case corresponding to the GREM. This means that both the Aizenman–Contucci stochastic stability and the Ghirlanda–Guerra identities, which were discovered in the setting of the SK model, also appear in the setting of the Ruelle probability cascades, suggesting some connection between the two.

The first partial answer to the above question was given in an influential work of Louis-Pierre Arguin and Michael Aizenman [5] who proved that, under a technical assumption that the overlap takes finitely many values in the thermodynamic limit, the Aizenman–Contucci stochastic stability implies the ultrametricity predicted by the Parisi ansatz. Soon after, it was shown in [43] that, under the same technical assumption, the Ghirlanda–Guerra identities also imply ultrametricity (an elementary proof can be found in [47]). Another approach was proposed by Talagrand in [65]. However, since at low temperature the overlap does not necessarily take finitely many values in the thermodynamic limit, all these results were not directly applicable to the SK model. Nevertheless, they strongly suggested that the stability properties can explain the Parisi ansatz and, indeed, the fact that the Ghirlanda–Guerra identities imply ultrametricity in general, without any technical assumptions, was proved in [50]. This means that the Ghirlanda–Guerra identities characterize the Ruelle probability cascades, which confirms the prediction of the physicists that the Gibbs measure in the SK model coincides with (or can be approximated by) the Ruelle probability cascades.

Even though the proof of this result, which will be given at the end of Chap. 2, is based only on the Ghirlanda–Guerra identities, it is important to mention that the Aizenman–Contucci stochastic stability played an important role in the discovery. It started with an observation made by Talagrand in 2007 (private communication, see also [66]) who noticed that in the setting of the Ruelle probability cascades the Ghirlanda–Guerra identities are contained in the Bolthausen–Sznitman invariance.

Talagrand's observation was reversed in [49] where the Ghirlanda–Guerra identities were combined with the Aizenman–Contucci stochastic stability and expressed as one unified stability property for the Gibbs measure in the SK model, which is the exact analogue of the Bolthausen–Sznitman invariance property in the setting of the Ruelle probability cascades. From this unified stability property one can derive a new invariance property that will appear in Sect. 2.5, where it will be used to prove the ultrametricity of the Gibbs measure in the SK model predicted by the Parisi ansatz. However, this new invariance property can be obtained much more easily as a direct consequence of the Ghirlanda–Guerra identities, which means that the Ghirlanda–Guerra identities alone explain the Parisi ansatz in the SK model and, for this reason, the Aizenman–Contucci stochastic stability will not be discussed in the book, even though behind the scenes it played a very important role. In some sense, this is good news because the Aizenman–Contucci stability is a more subtle property to work with than the Ghirlanda–Guerra identities, especially in the infinite-volume limit.

Once the structure of the Gibbs measure is understood, we will be in a position to prove the celebrated Parisi formula for the free energy. This will be the main focus of Chap. 3. The proof is based on two key results in the mathematical theory of the SK model—the replica symmetry breaking interpolation bound of Guerra [27] and the cavity computation scheme of Aizenman et al. [2]. The main idea of Guerra [27] can be viewed as a very clever interpolation between the SK model and the Ruelle probability cascades, which implies, due to monotonicity, that the Parisi formula is, in fact, an upper bound on the free energy of the SK model. Following this breakthrough discovery of Guerra, Talagrand proved in his famous tour-de-force paper [64] that the Parisi formula, indeed, gives the free energy in the SK model in the thermodynamic limit. Talagrand's ingenious proof finds a way around the Parisi ansatz for the Gibbs measure, but it is quite complicated. In Chap. 3 we will describe a much more direct approach to the matching lower bound based on the Aizenman–Sims–Starr cavity computation and the fact that the Gibbs measure can be approximated by the Ruelle probability cascades. Another advantage of this approach is that it yields the Parisi formula for all mixed p-spin models, while Talagrand's proof worked only for mixed p-spin models for even $p \geq 2$. For simplicity of notation, we only consider models without the external field, but all the results hold with obvious modifications in the presence of the external field.

In Chap. 4, we will study the Gibbs measure in the mixed p-spin models in more detail and describe the joint distribution of all spins in terms of the Ruelle probability cascades. This chapter is motivated by a different family of mean-field spin glass models that includes the random K-sat and diluted p-spin models, for which the main predictions of the physicists remain open and, since we can prove these predictions (in a certain sense) in the setting of the mixed p-spin models, we use it as an illustration of what is expected in these other models.

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