Applied Mathematical Sciences

Volume 183

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Mathematical Tools for the Study of the Incompressible Navier-Stokes Equations and Related Models

Springer

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ISSN 0066-5452
ISBN 978-1-4614-5974-3 ISBN 978-1-4614-5975-0 (eBook)
DOI 10.1007/978-1-4614-5975-0
Springer New York Heidelberg Dordrecht London

Library of Congress Control Number: 2012951655

Mathematics Subject Classification (2010): 35K55, 35J25, 35Q30, 76D03, 76D05, 76D07

Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.springer.com)

Preface

This monograph is a revised and augmented version of a previous book [24] written in French and published in 2006. Our objective is twofold. First, we want to introduce the reader to the modelling and the mathematical analysis in fluid mechanics. The central models we deal with are the incompressible Stokes and Navier–Stokes equations, whose derivation is exposed in the first chapter. Second, we introduce mathematical tools in a sufficiently general way so that the reader should be able to use them for studying many other kinds of nonlinear evolution partial differential equations.

In this spirit, we tried to make the book as self-contained as possible. The only prerequisites that are needed to begin reading are basic results in calculus, integration, and functional analysis. The second chapter is a recap of more or less standard analysis results that are used in this book. This includes for instance: the definitions and properties of weak and weak-$\star$ convergences in a Banach space, a short introduction to distribution theory, basic compactness results, the Bochner integral and related spaces, basic results in spectral theory, and so on.

In Chapter III, we introduce and study Sobolev spaces on Lipschitz domains of $\mathbb{R}^d$. The first section contains main definitions and properties of such domains. We describe, in particular, how to define the integral of functions defined on the boundary of such domains and we prove the fundamental Stokes formula. Sobolev spaces are then introduced and thoroughly studied in Section 2. An important role is played (in this chapter but also in the next ones) by suitable mollifying operators that permit us to prove the density of sets of smooth functions in various functional spaces. We also study embedding theorems, extension operators, trace and trace lifting operators, duality theory for Sobolev spaces, and the very important Poincaré and Hardy inequalities. The third section of the chapter is devoted to the introduction of suitable normal/tangential coordinates near the boundary of a sufficiently smooth domain. This framework is useful for the study of tangential Sobolev spaces and thus for the proof of regularity properties, up to the boundary, to solutions of elliptic problems (like the Stokes problem, for instance). All these notions are illustrated in the last section of the chapter where a complete study of the Laplace problem with Dirichlet or Neumann boundary conditions is given.

Chapter IV concerns the steady (incompressible) Stokes equations. This is a central topic for the study of any model in incompressible fluid mechanics. A basic tool in this theory is the Nečas inequality, a complete proof of which is given in Section 1. This inequality is used in the next section to study relations between the gradient fields and the divergence-free vector fields. This is fundamental in order to build the pressure in the Stokes system. The next two sections are dedicated to particular properties of the divergence and curl operators as well as some related function spaces. The analysis of the Stokes problem with Dirichlet boundary conditions is undertaken in Section 5, which includes the definition of the Stokes operator and application to the unsteady Stokes problem through the semigroup approach. We postponed to Section 6 the complete proof of elliptic regularity properties of the Stokes equations. The last three sections are concerned with some nonstandard variants of the Stokes problem, considering different kind of boundary or interface conditions.

The steady and unsteady Navier–Stokes equations are studied in Chapter V. For the unsteady problem, we prove existence and uniqueness (in 2D) of global weak solutions, we also investigate smoother solutions and discuss the parabolic regularisation properties of the system. Then, for the steady problem, we discuss existence and uniqueness issues in the case of homogeneous and nonhomogeneous boundary data and we deal with very basic stability issues for these solutions.

In Chapter VI, we study the most complex model considered in this book which is the one of the (unsteady) incompressible Navier–Stokes equations for a nonhomogeneous fluid. The first ingredient that is useful in the analysis of this problem is the study of weak solutions for the transport equation. We propose here, in the first section of this chapter, a self-contained exposition of the Di Perna–Lions theory of renormalized solutions for such equations including more recent developments related to the associated trace problem. Those results are then used to prove global existence of weak solutions for the initial- and boundary-value problem associated with the complete Navier–Stokes system in the second section of the chapter.

The final chapter is concerned with two different issues related to boundary conditions arising in numerical simulations of incompressible flows. The first section deals with the problem of outflow boundary conditions that one needs to choose in the case where the computational domain is, for practical reasons, smaller than the real physical domain. We analyse here a nonlinear boundary condition relating normal stress and momentum flux at the outflow boundary. In the second section of the chapter, we study the penalty method which can be used, for instance, to impose a homogeneous Dirichlet boundary condition on the boundary of an obstacle located in a computational domain whose geometry is simple. We prove convergence of the penalised solution towards the exact solution when the penalisation parameter goes to 0, through the description of a suitable boundary layer. Additionally to their practical interest, the analysis of these two problems gives the opportunity to introduce mathematical methods that the reader may find useful in other contexts.

Marseille,
August 2012,

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