

**Mathematical Statistics  
for Economics and Business**

**Second Edition**



**Ron C. Mittelhammer**

**Mathematical  
Statistics  
for Economics  
and Business**

**Second Edition**

With 93 Illustrations

 Springer

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ISBN 978-1-4614-5021-4      ISBN 978-1-4614-5022-1 (eBook)  
DOI 10.1007/978-1-4614-5022-1  
Springer New York Heidelberg Dordrecht London

Library of Congress Control Number: 2012950028

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Printed on acid-free paper

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*To my wife Linda,  
and to the memory of Carl, Edith, Dolly,  
and Ralph.*





# Preface to the Second Edition of *Mathematical Statistics for Economics and Business*



The general objectives of the second edition of *Mathematical Statistics for Economics and Business* remain the same as the first, namely, to provide a rigorous and accessible foundation in the principles of probability and in statistical estimation and inference concepts for beginning graduate students and advanced undergraduate students studying primarily in the fields of economics and business. Since its publication, the first edition of the book has found use by those from other disciplines as well, including the social sciences (e.g., psychology and sociology), applied mathematics, and statistics, even though many of the applied examples in later chapters have a decidedly “economics and business” feel (although the examples are chosen in such a way that they are fairly well “self-contained” and understandable for those who have not studied either discipline in substantial detail).

The general philosophy regarding how and why the book was originally written was presented in the preface to the first edition and in large measure could be inserted at this point for motivating the fundamental rationale for the second edition. This philosophy includes the necessity of having a conceptual base of probability and statistical theory to be able to fully understand the application and interpretation of applied econometric and business statistics methods, coupled with the need to have a treatment of the subject that, while rigorous, also assumes an accessible level of prerequisites that can be expected to have been met by a large majority of graduate students entering the fields. The choice of topic coverage is also deliberate and decidedly chosen to form the fundamental foundation on which econometric and business statistics methodology is built. With the ongoing expansion, in both scope and depth,

of econometric and statistical methodology for quantitative analyses in both economics and business, it has never been more important, and many are now thinking absolutely essential, that a base of formal probability and statistics understanding become part of student training to enable effective reading of the literature and success in the fields.

Regarding the nature of the updates and revisions that have been made in producing the second edition, many of the basic probability and statistical concepts remain in common with the first edition. The fundamental base of probability and statistics principles needed for later study of econometrics, business statistics, and a myriad of stochastic applications of economic and business theory largely intersects the topics covered in the first edition. While a few topics were deleted in the second edition as being less central to that foundation, many more have been added. These include the following: greater detail on the issue of parametric, semiparametric, and nonparametric models; an introduction to nonlinear least squares methods; Stieltjes integration has been added strategically in some contexts where continuous and discrete random variable properties could be clearly and efficiently motivated in parallel; additional testing methodology for the ubiquitous normality assumption; clearer differentiation of parametric and semiparametric testing of hypotheses; as well as many other refinements in topic coverage appropriate for applications in economics and business.

Perhaps the most important revision of the text has been in terms of the organization, exposition, and overall usability of the material. Reacting to the feedback of a host of professors, instructors, and individual readers of the first edition, the presentation of both the previous and new material has been notably reorganized and rewritten to make the text easier to study and teach from. At the highest level, the compartmentalization of topics is now better and easier to navigate through. All theorems and examples are now titled to provide a better foreshadowing of the content of the results and/or the nature of what is being illustrated. Some topics have been reordered to improve the flow of reading and understanding (i.e., the relatively more esoteric concept of events that cannot be assigned probability consistently has been moved to the end of a chapter and the review of elements of real analysis has been moved from the beginning of the asymptotic theory chapter to the appendix of the book), and in some cases, selected proofs of theorems that were essentially pure mathematics and that did little to bolster the understanding of statistical concepts were moved to chapter appendices to improve readability of the chapter text. A large number of new and expanded exercises/problems have been added to the chapters.

While a number of texts focused on statistical foundations of estimation and inference are available, *Mathematical Statistics for Economics and Business* is a text whose level of presentation, assumed prerequisites, examples and problems, and topic coverage will continue to provide a solid foundation for future study of econometrics, business statistics, and general stochastic economic and business theory and application. With its redesigned topic organization, additional topic coverage, revision of exposition, expanded set of problems, and continued focus on accessibility and motivation, the book will provide a conceptual foundation

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on which students can base their future study and understanding of rigorous econometric and statistical applications, and it can also serve as an accessible refresher for practicing professionals who wish to reestablish their understanding of the foundations on which all of econometrics, business statistics, and stochastic economic and business theory are based.

## Acknowledgments

In addition to all of the acknowledgments presented in the first edition, which certainly remain deserving of inclusion here, I would like to thank Ms. Danielle Engelhardt, whose enormous skills in typing, formatting, and proof-checking of the text material and whose always cheerful and positive “can-do” personality made the revision experience a much more enjoyable and efficient process. I am also indebted to Dr. Miguel Henry-Osorio for proofreading every character of every page of material and pointing out corrections, in addition to making some expositional suggestions that were very helpful to the revision process. Mr. Sherzod Akhundjanov also provided expert proof-checking, for which I am very grateful. I also thank Haylee and Hanna Gecas for their constant monitoring of my progress on the book revision and for making sure that I did not stray too far from the targeted timeline for the effort. I also wish to thank my colleague Dr. Tom Marsh, who utilized the first edition of this book for many years in the teaching of his econometrics classes and who provided me with helpful feedback on student learning from and topic coverage in the book. Finally, a deep thank you for the many comments and helpful suggestions I continued to receive over the years from my many doctoral students, the students who attended my statistics and econometrics classes here at the university; the many additional questions and comments I received from students elsewhere; and the input received from a host of individuals all over the world – the revision of the book has benefitted substantially from your input. Thank you all.





## Preface (First Edition)



This book is designed to provide beginning graduate students and advanced undergraduates with a rigorous and accessible foundation in the principles of probability and mathematical statistics underlying statistical inference in the fields of business and economics. The book assumes no prior knowledge of probability or statistics and effectively builds the subject “from the ground up.” Students who complete their studies of the topics in this text will have acquired the necessary background to achieve a mature and enduring understanding of statistical and econometric methods of inference and will be well equipped to read and comprehend graduate-level econometrics texts. Additionally, this text serves as an effective bridge to a more advanced study of both mathematical statistics and econometric theory and methods. The book will also be of interest to researchers who desire a decidedly business and economic-based treatment of the subject in terms of its topics, depth, breadth, examples, and problems.

Without the unifying foundations that come with training in probability and mathematical statistics, students in statistics and econometrics classes too often perceive the subject matter as a potpourri of formulae and techniques applied to a collection of special cases. The details of the cases and their solutions quickly fade for those who do not understand the reasons for using the procedures they attempt to apply. Many institutions now recognize the need for a more rigorous study of probability and mathematical statistics principles in order to prepare students for a higher-level, longer-lasting understanding of the statistical techniques employed in the fields of business and economics. Furthermore, quantitative analysis in these fields has progressed to the point where

a deeper understanding of the principles of probability and statistics is now virtually *necessary* for one to read and contribute successfully to quantitative research in economics and business. Contemporary students themselves know this and need little convincing from advisors that substantial statistical training must be acquired in order to compete successfully with their peers and to become effective researchers. Despite these observations, there are very few rigorous books on probability and mathematical statistics foundations that are also written with the needs of business and economics students in mind.

This book is the culmination of 15 years of teaching graduate level statistics and econometrics classes for students who are beginning graduate programs in business (primarily finance, marketing, accounting, and decision sciences), economics, and agricultural economics. When I originally took on the teaching assignment in this area, I cycled through a number of very good texts in mathematical statistics searching for an appropriate exposition for beginning graduate students. With the help of my students, I ultimately realized that the available textbook presentations were optimizing the wrong objective functions for our purposes! Some books were too elementary; other presentations did not cover multivariate topics in sufficient detail, and proofs of important results were omitted occasionally because they were “obvious” or “clear” or “beyond the scope of the text.” In most cases, they were neither obvious nor clear to students, and in many cases, useful and accessible proofs of the most important results can and should be provided at this level of instruction. Sufficient asymptotic theory was often lacking and/or tersely developed. At the extreme, material was presented in a sterile mathematical context at a level that was inaccessible to most beginning graduate students while nonetheless leaving notable gaps in topic coverage of particular interest to business and economics students. Noting these problems, gaps, and excesses, I began to teach the course from lecture notes that I had created and iteratively refined them as I interacted with scores of students who provided me with feedback regarding what was working—and what wasn’t—with regard to topics, proofs, problems, and exposition. I am deeply indebted to the hundreds of students who persevered through, and contributed to, the many revisions and continual sophistication of my notes. Their influence has had a substantial impact on the text: It is a time-tested and class-tested product. Other students at a similar stage of development should find it honest, accessible, and informative.

Instructors attempting to teach a rigorous course in mathematical statistics soon learn that the typical new graduate student in economics and business is thoroughly intelligent, but often lacks the sophisticated mathematical training that facilitates understanding and assimilation of the mathematical concepts involved in *mathematical* statistics. My experience has been that these students can understand and become functional with sophisticated concepts in mathematical statistics if their backgrounds are respected and the material is presented carefully and thoroughly, using a realistic level of mathematics. Furthermore, it has been my experience that most students are actually eager to see proofs of propositions, as opposed to merely accepting statements on faith, so long as the proofs do not insult the integrity of the nonmathematician.

Additionally, students almost always remark that the understanding and the long-term memory of a stated result are enhanced by first having worked through a formal proof of a proposition and then working through examples and problems that require the result to be applied.

With the preceding observations in mind, the prerequisites for the book include only the usual introductory college-level courses in basic calculus (including univariate integration and differentiation, partial differentiation, and multivariate integration of the iterated integral type) and basic matrix algebra. The text is largely self-contained for students with this preparation. A significant effort has been made to present proofs in ways that are accessible. Care has been taken to choose methods and types of proofs that exercise and extend the learning process regarding statistical results and concepts learned prior to the introduction of the proof. A generous number of examples are presented with a substantial amount of detail to illustrate the application of major theories, concepts, and methods. The problems at the end of the chapters are chosen to provide an additional perspective to the learning process. The majority of the problems are word problems designed to challenge the reader to become adept at what is generally the most difficult hurdle—translating descriptions of statistical problems arising in business and economic settings into a form that lends itself to solutions based on mathematical statistics principles. I have also warned students through the use of asterisks (\*) when a proof, concept, example, or problem may be stretching the bounds of the prerequisites so as not to frustrate the otherwise diligent reader, and to indicate when the help of an instructor or additional readings may be useful.

The book is designed to be versatile. The course that inspired this book is a semester-long four-credit intensive mathematical statistics foundation course. I do not lecture on all of the topics contained in the book in the 50 contact hours available in the semester. The topics that I do not cover are taught in the first half of a subsequent semester-long three-credit course in statistics and econometric methods. I have tended to treat Chapters 1–4 in detail, and I recommend that this material be thoroughly understood before venturing into the statistical inference portion of the book. Thereafter, the choice of topics is flexible. For example, the instructor can control the depth at which asymptotic theory is taught by her choice of whether the starred topics in Chapter 5 are discussed. While random sampling, empirical distribution functions, and sample moments should be covered in Chapter 6, the instructor has leeway in the degree of emphasis that she places on other topics in the chapter. Point estimation and hypothesis testing topics can then be mixed and matched with a minimal amount of back-referencing between the respective chapters.

Distinguishing features of this book include the care with which topics are introduced, motivated, and built upon one another; use of the appropriate level of mathematics; the generous level of detail provided in the proofs; and a familiar business and economics context for examples and problems. This text is bit longer than some of the others in the field. The additional length comes from additional explanation, and detail in examples, problems, and proofs, and not from a proliferation of topics which are merely surveyed rather than fully

developed. As I see it, a survey of statistical techniques is useful only after one has the fundamental statistical background to appreciate what is being surveyed. And this book provides the necessary background.

## Acknowledgments

I am indebted to a large number of people for their encouragement and comments. Millard Hastay, now retired from the Washington State University economics faculty, is largely responsible for my unwavering curiosity and enthusiasm for the field of theoretical and applied statistics and econometrics. George Judge has been a constant source of encouragement for the book project and over the years has provided me with very valuable and selfless advice and support in all endeavors in which our paths have crossed. I thank Jim Chalfant for giving earlier drafts of chapters a trial run at Berkeley, and for providing me with valuable student and instructor feedback. Thomas Severini at Northwestern provided important and helpful critiques of content and exposition. Martin Gilchrist at Springer-Verlag provided productive and pleasurable guidance to the writing and revision of the text. I also acknowledge the steadfast support of Washington State University in the pursuit of the writing of this book. Of the multitude of past students who contributed so much to the final product and that are too numerous to name explicitly, I owe a special measure of thanks to Don Blayney, now of the Economic Research Service, and Brett Crow, currently a promising Ph.D. candidate in economics at WSU, for reviewing drafts of the text literally character by character and demanding clarification in a number of proofs and examples. I also wish to thank many past secretaries who toiled faithfully on the book project. In particular, I wish to thank Brenda Campbell, who at times literally typed morning, noon, and night to bring the manuscript to completion, without whom completing the project would have been infinitely more difficult. Finally, I thank my wife Linda, who proofread many parts of the text, provided unwavering support, sustenance, and encouragement to me throughout the project, and despite all of the trials and tribulations, remains my best friend.



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