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VOLUME 29

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Nizar Touzi

# Optimal Stochastic Control, Stochastic Target Problems, and Backward SDE

With Chapter 13 by Agnès Tourin



The Fields Institute for Research  
in the Mathematical Sciences

 Springer

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ISSN 1069-5273                      ISSN 2194-3079 (electronic)  
ISBN 978-1-4614-4285-1            ISBN 978-1-4614-4286-8 (eBook)  
DOI 10.1007/978-1-4614-4286-8  
Springer New York Heidelberg Dordrecht London

Library of Congress Control Number: 2012943958

Mathematics Subject Classification (2010): 03C64, 14P15, 26A12, 26A93, 32C05, 32S65, 34C08, 34M40, 37S75, 58A17

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*Cover illustration:* Drawing of J.C. Fields by Keith Yeomans

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*I should like to express  
all my love to my family:  
Christine, our sons Ali and Héli, and our daughter Lilia,  
who accompanied me during this visit to Toronto,  
all my thanks to them for their patience while I was preparing  
these notes,  
and all my apologies for my absence even when I was  
physically present...*



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