The General Theory of Relativity

Anadijiban Das • Andrew DeBenedictis

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A Mathematical Exposition

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Anadijiban Das
Simon Fraser University
Burnaby, BC
Canada

Andrew DeBenedictis
Simon Fraser University
Burnaby, BC
Canada

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Dedicated to the memory of
Professor J. L. Synge

Preface

General relativity is to date the most successful theory of gravity. In this theory, the gravitational field is not a conventional force but instead is due to the geometric properties of a manifold commonly known as space–time. These properties give rise to a rich physical theory incorporating many areas of mathematics. In this vein, this book is well suited for the advanced mathematics or physics student, as well as researchers, and it is hoped that the balance of rigorous mathematics and physical insights and applications will benefit the intended audience. The main text and exercises have been designed both to gently introduce topics and to develop the framework to the point necessary for the practitioner in the field. This text tries to cover all of the important subjects in the field of classical general relativity in a mathematically precise way.

This is a subject which is often counterintuitive when first encountered. We have therefore provided extensive discussions and proofs to many statements, which may seem surprising at first glance. There are also many elegant results from theorems which are applicable to relativity theory which, if someone is aware of them, can save the individual practitioner much calculation (and time). We have tried to include many of them. We have tried to steer the middle ground between brute force and mathematical elegance in this text, as both approaches have their merits in certain situations. In doing this, we hope that the final result is "reader friendly." There are some sections that are considered advanced and can safely be skipped by those who are learning the subject for the first time. This is indicated in the introduction of those sections.

The mathematics of the theory of general relativity is mostly derived from tensor algebra and tensor analysis, and some background in these subjects, along with special relativity (relativity in the absence of gravity), is required. Therefore, in Chapter 1, we briefly provide the tensor analysis in Riemannian and pseudo-Riemannian differentiable manifolds. These topics are discussed in an arbitrary dimension and have many possible applications.

In Chapter 2, we review the special theory of relativity in the arena of the four-dimensional flat space–time manifold. Then, we introduce curved space–time and Einstein's field equations which govern gravitational phenomena.

In Chapter 3, we explore spherically symmetric solutions of Einstein's equations, which are useful, for example, in the study of nonrotating stars. Foremost among these solutions is the Schwarzschild metric, which describes the gravitational field outside such stars. This solution is the general relativistic analog of Newton's inverse-square force law of universal gravitation. The Schwarzschild metric, and perturbations of this solution, has been utilized for many experimental verifications of general relativity within the solar system. General solutions to the field equations under spherical symmetry are also derived, which have application in the study of both static and nonstatic stellar structure.

In Chapter 4, we deal with static and stationary solutions of the field equations, both in general and under the assumption of certain important symmetries. An important case which is examined at great length is the Kerr metric, which may describe the gravitational field outside of certain rotating bodies.

In Chapter 5, the fascinating topic of black holes is investigated. The two most important solutions, the Schwarzschild black hole and the axially symmetric Kerr black hole, are explored in great detail. The formation of black holes from gravitational collapse is also discussed.

In Chapter 6, physically significant cosmological models are pursued. (In this arena of the physical sciences, the impact of Einstein's theory is very deep and revolutionary indeed!) An introduction to higher dimensional gravity is also included in this chapter.

In Chapter 7, the mathematical topics regarding Petrov's algebraic classification of the Riemann and the conformal tensor are studied. Moreover, the Newman–Penrose versions of Einstein's field equations, incorporating Petrov's classification, are explored. This is done in great detail, as it is a difficult topic and we feel that detailed derivations of some of the equations are useful.

In Chapter 8, we introduce the coupled Einstein–Maxwell–Klein–Gordon field equations. This complicated system of equations classically describes the self-gravitation of charged scalar wave fields. In the special arena of spherically symmetric, static space–time, these field equations, with suitable boundary conditions, yield a nonlinear eigenvalue problem for the allowed theoretical charges of gravitationally bound wave-mechanical condensates.

Eight appendices are also provided that deal with special topics in classical general relativity as well as some necessary background mathematics.

The notation used in this book is as follows: The Roman letters i, j, k, l, m, n, etc. are used to denote subscripts and superscripts (i.e., covariant and contravariant indices) of a tensor field's components relative to a coordinate basis and span the full dimensionality of the manifold. However, we employ parentheses around the letters (a), (b), (c), (d), (e), (f), etc. to indicate components of a tensor field relative to an orthonormal basis. Greek indices are used to denote components that only span the dimensionality of a hypersurface. In our discussions of space–time, these Greek indices indicate spatial components only. The flat Minkowskian metric tensor components are denoted by d_{ij} or $d_{(a)(b)}$. Numerically they are the same, but conceptually there is a subtle difference. The signature of the space–time metric is

$+2$ and the conventions for the definitions of the Riemann, Ricci, and conformal tensors follow the classic book of Eisenhart.

We would like to thank many people for various reasons. As there are so many who we are indebted to, we can only explicitly thank a few here, in the hope that it is understood that there are many others who have indirectly contributed to this book in many, sometimes subtle, ways.

I (A. Das) learned much of general relativity from the late Professors J. L. Synge and C. Lanczos during my stay at the Dublin Institute for Advanced Studies. Before that period, I had as mentors in relativity theory Professors S. N. Bose (of Bose–Einstein statistics), S. D. Majumdar, and A. K. Raychaudhuri in Kolkata. During my stay in Pittsburgh, I regularly participated in, and benefited from, seminars organized by Professor E. Newman. In Canada, I had informal discussions with Professors F. Cooperstock, J. Gegenberg, W. Israel, and E. Pechlaner and Drs. P Agrawal, S. Kloster, M. M. Som, M. Suvegas, and N. Tariq. Moreover, in many international conferences on general relativity and gravitation, I had informal discussions with many adept participants through the years.

I taught the theory of relativity at University College of Dublin, Jadavpur University (Kolkata), Carnegie-Mellon University, and mostly at Simon Fraser University (Canada). Stimulations received from the inquiring minds of students, both graduate and undergraduate, certainly consolidated my understanding of this subject.

Finally, I thank my wife, Mrs. Purabi Das. I am very grateful for her constant encouragement and patience.

I (A. DeBenedictis) would like to thank all of the professors, colleagues, and students who have taught and influenced me. As mentioned previously, there are far too many to name them all individually. I would like to thank Professor E. N. Glass of the University of Michigan-Ann Arbor and the University of Windsor, who gave me my first proper introduction to this fascinating field of physics and mathematics. I would like to thank Professor K. S. Viswanathan of Simon Fraser University, from whom I learned, among the many things he taught me, that this field has consequences in theoretical physics far beyond what I originally had thought.

I would also like to thank my colleagues whom I have met over the years at various institutions and conferences. All of them have helped me, even if they do not know it. Discussions with them, and their hospitality during my visits, are worthy of great thanks. During the production of this work, I was especially indebted to my colleagues in quantum gravity. They have given me the appreciation of how difficult it is to turn the subject matter of this book into a quantum theory, and opened up a fascinating new area of research to me. The quantization of the gravitational field is likely to be one of the deepest, difficult, and most interesting puzzles in theoretical physics for some time. I hope that this text will provide a solid background for half of that puzzle to those who choose to tread down this path.

I would also like to thank the students whom I have taught, or perhaps they have taught me. Whether it be freshman level or advanced graduate level, I can honestly say that I have learned something from every class that I have taught.

Not least, I extend my deepest thanks and appreciation to my wife Jennifer for her encouragement throughout this project. I do not know how she did it.

We both extend great thanks to Mrs. Sabine Lebhart for her excellent and timely typesetting of a very difficult manuscript.

Finally, we wish the best to all students, researchers, and curious minds who will each in their own way advance the field of gravitation and convey this beautiful subject to future generations. We hope that this book will prove useful to them.

Vancouver, Canada

Anadijiban Das
Andrew DeBenedictis

Contents

List of Figures

List of Tables

Symbols[1], [2]

$\square$	d'Alembertian operator, completion of an example
$\blacksquare$	Q.E.D., completion of proof
$\cdot$	Central dot denotes multiplication (used to make crowded equations more readable)
$\equiv$	Identity
$:=$, $=:$	Definition, which is an identity involving new notation
$*$	Hodge star operation, tortoise coordinate designation
$\vert_{..}$	Constrained to a curve or surface
$\in$	Belongs to
${}^{r}_{s}\mathbf{0}$, $\mathbf{O}^{\cdot}_{\cdot\cdot\cdot}$	$(r+s)$th order zero tensor. (In the latter the number of dots indicate r and s.)
$\vec{\mathbf{O}}$	Zero vector
$[jk, i]$	Christoffel symbol of the 1st kind
$\left\{ {i \atop j\ k} \right\}$	Christoffel symbol of the 2nd kind
$[c \leftrightarrow d]$, $\{\mu \leftrightarrow \nu\}$, etc.	Represents the previous term in brackets of an expression but with the given indices interchanged
a	Angular momentum parameter, expansion factor in F–L–R–W metric
$\Vert\vec{\mathbf{A}}\Vert$	Norm or length of a vector
$A \cup B$	Union of two sets
$A \cap B$	Intersection of two sets
$A \times B$	Cartesian product of two sets

[1]For common tensors, only the coordinate component form is shown in this list.

[2]Occasionally the symbols listed here will also have other definitions in the text. We tabulate the most common definitions here as it should be clear in the text where the meanings differ from those in this list.

$A \subset B$	A is a subset of B
$(\vec{\mathbf{A}} \cdot \vec{\mathbf{B}})_g \equiv \mathbf{g}..(\vec{\mathbf{A}}, \vec{\mathbf{B}})$	Inner product between two vectors
${}_p\mathbf{A} \wedge {}_q\mathbf{B}$	Wedge product between a p-form and a q-form
$[\vec{\mathbf{A}}, \vec{\mathbf{B}}] := \vec{\mathbf{A}}\,\vec{\mathbf{B}} - \vec{\mathbf{B}}\,\vec{\mathbf{A}}$	Lie bracket or commutator
$\mathcal{A}$	Electric potential, (also a function used in five-dimensional cosmologies)
A^i	Components of the electromagnetic 4-potential
α	Affine parameter for a null geodesic, Newman–Penrose spin-coefficient
$\mathcal{B}$	Magnetic potential, bivector set
β	Expansion coefficient for 5th dimension, Newman–Penrose spin-coefficient
c	Speed of light (usually set to 1)
$\mathcal{C}$	Conformal group, causality violating region
C^r	Differentiability class r
$\mathcal{C}^i$	Coordinate conditions
${}^u_v\mathcal{C}[{}^r_s\mathbf{T}]$	Contraction operation of a tensor field ${}^r_s\mathbf{T}$
$C^i{}_{jkl}$	Components of Weyl's conformal tensor
$\mathbb{C}$	The set of all complex numbers
(χ, U)	Coordinate chart for a differentiable manifold
$\chi(p) = x$	Local coordinates of a point p in a manifold.
$\equiv (x^1, x^2, \ldots, x^N)$	In some places $x \in \mathbb{R}$.
χ_{ij}	Extended extrinsic curvature components
D	A domain in $\mathbb{R}^N$ (open and connected)
D_i	Gauge covariant derivative
∂D	$(N-1)$-dimensional boundary of D
∇_i	Covariant derivatives
$\dfrac{D}{\partial t}$	Covariant derivative along a curve
Δ	Laplacian in a manifold with metric, determinant of Γ_{ij}
∇^2	Laplacian in a Euclidean space
δ^i_j	Components of Kronecker delta (or identity matrix)
${}^p_p\delta,\ \delta^{i_1,\ldots,i_p}{}_{j_1,\ldots,j_p}$	Generalized Kronecker delta
$\mathrm{d}f,\ \mathrm{d}[{}_p\mathbf{W}]$	Exterior derivative of f or ${}_p\mathbf{W}$
d_{ij}	Components of flat space metric
$\dfrac{\partial(\widehat{x}^1, \ldots, \widehat{x}^N)}{\partial(x^1, \ldots, x^N)}$	Jacobian of a coordinate transformation
e	Electric charge, exponential
$\mathcal{E}_{ij},\ \tilde{\mathcal{E}}_{ij},\ \mathcal{E}^l{}_{ijk}$	Components of Einstein equations (in various forms)
$\vec{\mathbf{E}},\ E_\alpha$	Electric field and its components
$\mathbb{E}_N$	N-dimensional Euclidean space
$\{\vec{\mathbf{e}}_{(a)}\}_1^N$ $:= \{\vec{\mathbf{e}}_{(1)}, \ldots, \vec{\mathbf{e}}_{(N)}\}$	A basis set for a vector space

$\{\vec{\mathbf{E}}_{(a)}\}_1^4 := \left\{\vec{\mathbf{m}}, \vec{\bar{\mathbf{m}}}, \vec{\mathbf{l}}, \vec{\mathbf{k}}\right\}$	A complex null basis set for Newman–Penrose formalism
ε	A small number, Newman–Penrose spin coefficient, coefficient of a perturbation
$\varepsilon_{i_1 i_2,\ldots,i_N}$	Totally antisymmetric permutation symbol (Levi-Civita)
η^i	Components of the geodesic deviation vector
$\eta^{(a)(b)}$	Components of metric tensor relative to a complex null tetrad
$\eta_{i_1,i_2,\ldots,i_N}$	Totally antisymmetric pseudo (or oriented) tensor (Levi-Civita)
f^α	Newtonian force
$f_{ij}(x,u)$	Finsler metric components
F^i	4-force components
F_{ij}	Electromagnetic tensor field tensor components
g, $\lvert g\rvert$	Metric tensor determinant and its absolute value
G	Gravitational constant (usually set to 1)
g_{ij}	Metric tensor components
G^i_j	Einstein tensor components
γ	A parametrized curve into a manifold, Newman–Penrose spin coefficient
$\mathcal{X} := \chi \circ \gamma$	A parametrized curve into $\mathbb{R}^N$
Γ	The image of a parametrized curve into $\mathbb{R}^N$, characteristic matrix
Γ_{ij}	Characteristic matrix components
$\mathbb{\Gamma}_{(a)(b)(c)}$	Complex Ricci rotation coefficients
$\gamma_{(a)(b)(c)}$	Ricci rotation coefficients
$\gamma^k{}_{ij}$	Independent connection components in Hilbert-Palatini variational approach
$\hbar$	Reduced Planck's constant (usually set to 1)
h^i, h^{ij}, h^k_{ij}	Variations of vector, second-rank tensor, Christoffel connection respectively
$\vec{\mathbf{H}}$, H_α	Magnetic field and its components
$\mathcal{H}$	Relativistic Hamiltonian
$\mathbf{I}^{\cdot}_{\cdot}$	Identity tensor
J	Action functional or action integral
J^i	4-current components
J^{ik}	Total angular momentum components
$\vec{\mathbf{k}}$	Real null tetrad vector
k^i	Wave vector (or number) components
k_0	Curvature of spatial sections of F–L–R–W metric
$K(u)$	Gaussian curvature
$\vec{\mathbf{K}}$, K_i	A Killing vector and corresponding components
$K_{\mu\nu}$	Extrinsic curvature components of a hypersurface

κ	Einstein equation constant ($= 8\pi G/c^4$ in common units)
$\kappa_{(A)}, \kappa_{(0)}$	A^{th} curvature, Newman–Penrose spin coefficient
$\vec{\mathbf{l}}$	Real null tetrad vector
$l^i{}_j, L^i{}_j$	Components of a generalized Lorentz transformation
$[L]^{\mathrm{T}}$	Transposed matrix
L	A Lagrangian function
$L_{\vec{\mathbf{v}}}$	Lie derivative
$\mathcal{L}$	Lagrangian density
$\mathcal{L}_{(\mathrm{I})}$	Lagrangian function from super-Hamiltonian
λ	Eigenvalue, Lagrange multiplier, electromagnetic gauge function, Newman–Penrose spin coefficient
$\lambda_{(i)}$	ith eigenvalue
$\vec{\lambda}_{(A)}(s)$	Ath normal vector to a curve
$\lambda^i{}_{(a)}, \mu^{(a)}{}_i$	Components of orthonormal basis
Λ	Cosmological constant
$m, M(s)$	Mass, mass function
$\vec{\mathbf{m}}, \vec{\overline{\mathbf{m}}}$	Complex null tetrad vectors
M, M_N	A differentiable manifold, N dimensional differentiable manifold
M	"Total mass" of the universe
$\mathcal{M}^i, {}^*\mathcal{M}^i$	Maxwell vector (and dual) components
μ	Mass density, Newman–Penrose spin coefficient
N	Dimension of tangent vector space, lapse function in A.D.M. formalism
n^i	Unit normal vector components
N^α	Shift vector in A.D.M. formalism
ν	Frequency, Newman–Penrose spin coefficient
$O(p, n; \mathbb{R})$	Generalized Lorentz group
$\mathcal{I}O(p, n; \mathbb{R})$	Generalized Poincaré group
p	Point in a manifold, polynomial equation, pressure
$p^{\#}$	Polynomial equation for invariant eigenvalues
$p_{\parallel}, p_{\perp}$	Parallel pressure and transverse pressure respectively
$p^i, \mathcal{P}^i$	4-momentum components
$\pi_{(0)}$	Newman–Penrose spin coefficient
π^k	Projection mapping
$\mathcal{P}^i{}_j$	Projection tensor field components
ϕ	Characteristic surface function of a p.d.e., scalar field
Φ	Born-Infeld (or tachyonic) scalar field, (also a function used in five-dimensional cosmologies)
$\phi^\alpha_{(\mathrm{ext})}$	External force density
φ^{ij}	Complex electromagnetic field tensor components
$\Phi_{(A)(B)}$	Complex Ricci components ($A,\ B \in \{0,\ 1,\ 2\}$)
$\Phi^{i_1,\ldots,i_r}{}_{j_1,\ldots,j_s}$	Components of an oriented, relative tensor field of weight w

Ψ	Complex Klein-Gordon field
$\Psi_{(J)}$	Complex Jth Weyl components ($J \in \{0, \ldots, 4\}$)
$Q_{(a)(d)}$	Complex Weyl tensor with second and third index projected in a timelike direction
R	Ricci curvature scalar (or invariant)
R_{ij}	Components of Ricci tensor
$R^{i}{}_{jk}$	Components of Cotton–Schouten–York tensor
$R^{i}{}_{jkl}$	Components of Riemann-Christoffel tensor
$\mathbb{R}$	The set of real numbers, complex Ricci scalar
$\mathbb{R}^N := \underbrace{\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}}_{N}$	Cartesian product of N copies of the set $\mathbb{R}$
ρ	Mass density, proper energy density, Newman–Penrose spin coefficient
s^{ij}, S^{ij}	Components of relativistic stress tensor (special and general respectively)
S_{ijkl}	Components of symmetrized curvature tensor
s	Arc separation parameter
S^2	Two-dimensional spherical surface
σ	Electrical charge density, Newman–Penrose spin coefficient, separation of a vector field, Klein-Gordon equation
$\sigma^{\alpha\beta}, \sigma^{ij}$	Stress density, shear tensor components
$\sum$, Σ	Arc separation function, function in Kerr metric, summation
$\vec{\mathbf{t}}_x$	Tangent vector of the image Γ at the point x
T_x	Tangent vector space of a manifold
$\tilde{T}_x$	Cotangent (or dual) vector space of a manifold
T^{ij}	Components of energy–momentum–stress tensor
$\mathbf{T}^{\cdot}_{\cdot\cdot}$, $T^{i}{}_{jk}$	Torsion tensor and the corresponding components
${}^{r}_{s}\mathbf{T}$	Tensor field of order $(r+s)$
$T^{i_1,\ldots,i_r}{}_{j_1,\ldots,j_s}$	Coordinate components of the (same) tensor field
$T^{(a_1),\ldots,(a_r)}{}_{(b_1),\ldots,(b_s)}$	Orthonormal components of the (same) tensor field
$\mathbb{T}^{(a_1),\ldots,(a_r)}{}_{(b_1),\ldots,(b_s)}$	Complex tensor field components
${}^{r}_{s}\mathbf{T} \otimes {}^{p}_{q}\mathbf{S}$	Tensor (or outer product) of two tensor fields
$\mathcal{T}^{i}$	Conservation law components
τ	Affine parameter along geodesic (usu. proper time), Newman–Penrose spin coefficient
${}^{r}_{s}\mathcal{T}(T_x(\mathbb{R}^N))$	Tensor bundle
Θ_{ij}	Expansion tensor components, T-domain energy–momentum–stress tensor
${}^{r}_{s}\Theta$	Components of a relative tensor field
U	an open subset of a manifold
$\mathsf{U}_{(a)(b)}$, $\mathsf{V}_{(a)(b)}$, $\mathsf{W}_{(a)(b)}$	Components of complex bivector fields (see definitions (7.48i–vi))

u^i, U^i, $\mathcal{U}^i$	4-velocity components
$V^\alpha(t)$, $\mathcal{V}^\alpha$	Newtonian or Galilean velocity
W, w	Effective Newtonian potential
W	Lambert's W-function, (symbol also used for other functions in axi-symmetric metrics)
$\mathcal{W}$	Work function
${}_p\mathbf{W}$, $W_{i_1,\ldots,i_p}$	p-form and its antisymmetric components
ω_{ij}	Vorticity tensor components
Ω	Synge's world function
$x = \mathcal{X}(t)$, $x^i = \mathcal{X}^i(t)$	A parametrized curve in $\mathbb{R}^N$
$x = \xi(u)$, $x^i = \xi^i(u^1, \ldots, u^D)$	A parametrized submanifold
Y	Coefficient of spherical line element in Tolman-Bondi coordinates
z, $\bar{z}$	A complex variable and its conjugate
$\mathbb{Z}$, $\mathbb{Z}^+$	The set of integers, the set of positive integers