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Coxeter Graphs and Towers of Algebras

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PREFACE

The paper [Jo1] on subfactors of von Neumann factors has stimulated much research in von Neumann algebras. Quite unexpectedly, it was discovered soon after the appearance of [Jo1] that certain algebras which are used there for the analysis of subfactors could also be used to define a new polynomial invariant for links [Jo3]. The period of activity following this discovery saw the creation of a number of related invariants as well as the successful use of these invariants in knot theory. Furthermore, recent effort to understand the fundamental nature of the the new link invariants has led to connections with invariant theory, statistical mechanics and quantum field theory. In turn the link invariants, the notion of a quantum group, and the quantum Yang-Baxter equation have had great impact on the study of subfactors.

It is not yet the time to give a comprehensive account of these developments, and we make no attempt to do so here. Our subject is certain algebraic and von Neumann algebraic topics closely related to the original paper [Jo1]. However, in order to promote, in a modest way, the contact between diverse fields of mathematics, we have tried to make this work accessible to the broadest audience. Consequently, this book contains much elementary expository material.

We give here a brief preview of the book. Each of the four chapters has its own introduction, with a more thorough description of the contents.

Chapter 1 begins with a (slightly new guise of) a familiar combinatorial problem: to classify finite matrices over the non-negative integers which have Euclidean norm no greater than 2. These are classified by the ubiquitous *Coxeter graphs* of type A, D, or E (see [HHSV] for other occurrences of these graphs) and the set of possible norms is $\{2\} \cup \{2\cos\pi/q : q \geq 2\}$.

The central theme of the book — the discussion of which begins in Chapter 2 — is the tower of algebras $M_0 \subset M_1 \subset \cdots \subset M_k \subset \cdots$ determined by a pair $M_0 \subset M_1$ of algebras (with the same identity). The tower can be used to define various invariants of the pair, including the index $[M_1:M_0]$

In Chapters 2 and 3, we study two cases of the tower construction in detail.

In Chapter 2, the algebras are finite direct sums of full matrix algebras over some field. A pair $M_0 \subset M_1$ is described, up to isomorphism, by an inclusion matrix Λ with non-negative integer entries. This matrix may be encoded as a graph, known as the Bratteli diagram of the pair. It turns out that the index $[M_1:M_0]$ equals $\|\Lambda\|^2$; thus it follows from Chapter 1 that $[M_1:M_0] \leq 4$ if and only if the Bratteli diagram is a Coxeter graph of type A, D, or E.

In Chapter 3, the algebras are finite von Neumann algebras with finite dimensional centers. Somewhat surprisingly, the results of Chapter 2 essentially extend to this setting. But now a pair $M_0 \subset M_1$ is (partially) described by an inclusion matrix Λ with entries in $\{2\cos\pi/q : q \geq 2\} \cup \{r : r \geq 2\}$, and pairs with index no greater than 4 are associated to Coxeter graphs of arbitrary type, including types B, F, G, H, I.

Finally, Chapter 4 is a further analysis of pairs $N \subset M$ of finite factors of finite index. There are two main themes. The first is the notion of a commuting square, due to Popa [Pop1], and its use in approximating pairs of hyperfinite II_1 factors simultaneously by finite dimensional von Neumann algebras. The second theme is the derived tower of a pair of II_1 factors, which is the chain of (necessarily finite dimensional) relative commutants $M_0' \cap M_k$ in the tower. All the information in the derived tower can be encoded in a (possibly infinite) graph, the principal graph of the pair. When the index is less than 4, the graph is a Coxeter graph of type A, D, or E.

In Chapter 2, we also describe how a certain quotients of the Hecke algebra of type A appear in the tower construction associated to a pair $M_0 \subset M_1$ of, say, finite dimensional semi-simple algebras over the field $\mathbb{C}$. For each choice of a positive faithful trace tr on M_1 , there is a unique trace preserving conditional expectation $E_1 : M_1 \rightarrow M_0$, and it turns out that M_2 is naturally generated by M_1 and E_1 . Now if the trace tr satisfies the so-called Markov condition, then the situation propagates up the tower, and each algebra M_{k+1} is naturally generated by M_k and a conditional expectation $E_k : M_k \rightarrow M_{k-1}$, for all $k \geq 1$. Moreover the E_k 's are idempotents which satisfy the "braiding" relations

$$\begin{aligned} \beta E_i E_j E_i &= E_i & \text{if } |i - j| = 1, \text{ and} \\ E_i E_j &= E_j E_i & \text{if } |i - j| \geq 2, \end{aligned}$$

where $\beta = [M_1 : M_0]$. The abstract algebra $\mathcal{A}_{\beta,k}$ presented by generators $\{1, E_1, \dots, E_{k-1}\}$ and relations as above is a quotient of the Hecke algebra $H_k(q)$, where $q \in \mathbb{C}$ satisfies $\beta = 2 + q + q^{-1}$.

Although we do not discuss this in the text, we might remark here that the map

$$\mathbf{B}_\infty \longrightarrow H_\infty(q) \longrightarrow \text{alg}\{1, E_1, \dots\} \xrightarrow{\text{tr}} \mathbb{C}$$

where $\mathbf{B}_\infty$ is the inductive limit of the braid groups $\mathbf{B}_k$, is, up to a normalization, the Jones link invariant [Jo3]. Also let us point out that to obtain the Jones invariant in this way, it is necessary to deal only with finite dimensional algebras, not the less familiar infinite dimensional von Neumann factors.

There are several appendices.

Appendix I extends the computations of Chapter 1.

Appendix II.a relates complex semi-simple algebras and finite dimensional C^* -algebras. Appendix II.b explains one appearance of the algebras $\mathcal{A}_{\beta,k}$ in statistical mechanics. Appendix II.c is a further discussion of $\mathcal{A}_{\beta,k}$ for special values of β .

Appendix III is an exposition of Hecke subgroups in $PSL_2(\mathbb{R})$, and thus another famous occurrence of the sequence $(2\cos(\pi/k))_{k \geq 3}$.

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