

# Universitext

*Editors*

F.W. Gehring  
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## Universitext

Editors: F.W. Gehring, P.R. Halmos

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K.T. Smith

Power Series  
from a Computational Point  
of View



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## PREFACE

At the end of the typical one quarter course on power series the students lack the means to decide whether  $1/(1+x^2)$  has an expansion around any point  $\neq 0$ , or the tangent has an expansion anywhere - and the means to evaluate and predict errors.

In using power series for computation the main problems are: 1) To predict a priori the number  $N$  of terms needed to do the computation with a specified accuracy; and 2) To find the coefficients  $a_0, \dots, a_N$ . These are the problems addressed in the book.

Typical computations envisioned are:  
calculate with error  $\leq 10^{-6}$  the integrals

$$\int_0^{\pi/2} (\pi/2-x)\tan x \, dx \quad , \quad \int_0^1 x(1+e^x)^{1/2} \, dx,$$

or the solution to the differential equation

$$y''+(\sin x)y'+x^2y = 0, \quad y(0) = 0, \quad y'(0) = 1,$$

on the interval  $0 \leq x \leq 1$ .

This computational point of view may seem narrow, but, in fact, such computations require the understanding and use of many of the important theorems of elementary analytic function theory: Cauchy's Integral Theorem, Cauchy's Inequalities, Unique Continuation, Analytic Continuation and the Monodromy Theorem, etc. The computations provide an effective motivation for learning the theorems and a sound basis for understanding them. To other scientists the rationale for the

computational point of view might be the need for efficient accurate calculation; to mathematicians it is the motivation for learning theorems and the practice with inequalities,  $\epsilon$ 's,  $\delta$ 's, and  $N$ 's.

Throughout the book  $\epsilon = 10^{-6}$ . Experience shows that  $10^{-6}$  (or any other specific small number) is more acceptable and challenging to students than a vague and mysterious  $\epsilon$ , while, of course, there is no difference in the mathematical analysis.  $10^{-6}$  is chosen so that those who want to can perform realistic computations on a 16 bit microcomputer. The computer code is usually a mathematical proof in a disguise that is appealing to students, and it is strongly recommended as a required part of the problem solutions, simply as a learning device.

Since the book contains complete proofs of the theorems cited above, it is clear that the whole cannot be covered in one quarter. A one quarter course, especially one for engineers, physicists, etc., might cover Chapters 1 and 2 with intensive discussion of the meaning and application of the theorems, but without proofs. (This has been done several times with gratifying success.) A two or three quarter course might cover the whole with proofs and other topics. (The simple proof of the general homotopy version of Cauchy's Theorem was devised in such a course about twenty-five years ago.)

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