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Intermediate Real Analysis

With 100 Illustrations



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To
Wilhelm Magnus

Contents

Preface	xii
Chapter I	
Preliminaries	1
1. Sets	1
2. The Set \mathbb{R} of Real Numbers	4
3. Some Inequalities	10
4. Interval Sets, Unions, Intersections, and Differences of Sets	11
5. The Non-negative Integers	16
6. The Integers	21
7. The Rational Numbers	23
8. Boundedness: The Axiom of Completeness	24
9. Archimedean Property	26
10. Euclid's Theorem and Some of Its Consequences	28
11. Irrational Numbers	34
12. The Noncompleteness of the Rational Number System	38
13. Absolute Value	39
Chapter II	
Functions	42
1. Cartesian Product	42
2. Functions	43
3. Sequences of Elements of a Set	47
4. General Sums and Products	51
5. Bernoulli's and Related Inequalities	55
6. Factorials	57
7. Onto Functions. n th Root of a Positive Real Number	61
8. Polynomials. Certain Irrational Numbers	64
9. One-to-One Functions. Monotonic Functions	65
10. Composites of Functions. One-to-One Correspondences. Inverses of Functions	68
11. Rational Exponents	75
12. Some Inequalities	78

Chapter III	
Real Sequences and Their Limits	92
1. Partially and Linearly Ordered Sets	92
2. The Extended Real Number System \mathbb{R}^*	93
3. Limit Superior and Limit Inferior of Real Sequences	97
4. Limits of Real Sequences	100
5. The Real Number e	107
6. Criteria for Numbers To Be Limits Superior or Inferior of Real Sequences	110
7. Algebra of Limits: Sums and Differences of Sequences	119
8. Algebra of Limits: Products and Quotients of Sequences	127
9. L'Hôpital's Theorem for Real Sequences	139
10. Criteria for the Convergence of Real Sequences	142
Chapter IV	
Infinite Series of Real Numbers	151
1. Infinite Series of Real Numbers. Convergence and Divergence	151
2. Alternating Series	155
3. Series Whose Terms Are Nonnegative	158
4. Comparison Tests for Series Having Nonnegative Terms	162
5. Ratio and Root Tests	166
6. Kummer's and Raabe's Tests	173
7. The Product of Infinite Series	176
8. The Sine and Cosine Functions	182
9. Rearrangements of Infinite Series and Absolute Convergence	189
10. Real Exponents	196
Chapter V	
Limits of Functions	202
1. Convex Set of Real Numbers	202
2. Some Real-Valued Functions of a Real Variable	203
3. Neighborhoods of a Point. Accumulation Point of a Set	207
4. Limits of Functions	212
5. One-Sided Limits	224
6. Theorems on Limits of Functions	228
7. Some Special Limits	232
8. $P(x)$ as $x \rightarrow \pm \infty$, Where P is a Polynomial on \mathbb{R}	236
9. Two Theorems on Limits of Functions. Cauchy Criterion for Functions	238
Chapter VI	
Continuous Functions	240
1. Definitions	240
2. One-Sided Continuity. Points of Discontinuity	246
3. Theorems on Local Continuity	250
4. The Intermediate-Value Theorem	252
5. The Natural Logarithm: Logs to Any Base	258
6. Bolzano–Weierstrass Theorem and Some Consequences	266

7. Open Sets in \mathbb{R}	270
8. Functions Continuous on Bounded Closed Sets	273
9. Monotonic Functions. Inverses of Functions	275
10. Inverses of the Hyperbolic Functions	281
11. Uniform Continuity	283
Chapter VII	
Derivatives	290
1. The Derivative of a Function	290
2. Continuity and Differentiability. Extended Differentiability	296
3. Evaluating Derivatives. Chain Rule	300
4. Higher-Order Derivatives	306
5. Mean-Value Theorems	316
6. Some Consequences of the Mean-Value Theorems	322
7. Applications of the Mean-Value Theorem. Euler's Constant	329
8. An Application of Rolle's Theorem to Legendre Polynomials	336
Chapter VIII	
Convex Functions	340
1. Geometric Terminology	340
2. Convexity and Differentiability	347
3. Inflection Points	358
4. Trigonometric Functions	360
5. Some Remarks on Differentiability	365
6. Inverses of Trigonometric Functions. Tschebyscheff Polynomials	368
7. Log Convexity	376
Chapter IX	
L'Hôpital's Rule—Taylor's Theorem	378
1. Cauchy's Mean-Value Theorem	378
2. An Application to Means and Sums of Order t	386
3. The O - o Notation for Functions	391
4. Taylor's Theorem of Order n	394
5. Taylor and Maclaurin Series	400
6. The Binomial Series	406
7. Tests for Maxima and Minima	412
8. The Gamma Function	417
9. Log-Convexity and the Functional Equation for Γ	423
Chapter X	
The Complex Numbers. Trigonometric Sums. Infinite Products	427
1. Introduction	427
2. The Complex Number System	427
3. Polar Form of a Complex Number	437
4. The Exponential Function on \mathbb{C}	444
5. n th Roots of a Complex Number. Trigonometric Functions on \mathbb{C}	450
6. Evaluation of Certain Trigonometric Sums	456

7. Convergence and Divergence of Infinite Products	465
8. Absolute Convergence of Infinite Products	474
9. Sine and Cosine as Infinite Products. Wallis' Product. Stirling's Formula	478
10. Some Special Limits. Stirling's Formula	483
11. Evaluation of Certain Constants Associated with the Gamma Function	491
Chapter XI	
More on Series: Sequences and Series of Functions	494
1. Introduction	494
2. Cauchy's Condensation Test	495
3. Gauss' Test	498
4. Pointwise and Uniform Convergence	505
5. Applications to Power Series	513
6. A Continuous But Nowhere Differentiable Function	521
7. The Weierstrass Approximation Theorem	525
8. Uniform Convergence and Differentiability	534
9. Application to Power Series	545
10. Analyticity in a Neighborhood of x_0 . Criteria for Real Analyticity	554
Chapter XII	
Sequences and Series of Functions II	558
1. Arithmetic Operations with Power Series	558
2. Bernoulli Numbers	565
3. An Application of Bernoulli Numbers	575
4. Infinite Series of Analytic Functions	579
5. Abel's Summation Formula and Some of Its Consequences	587
6. More Tests for Uniform Convergence	593
Chapter XIII	
The Riemann Integral I	601
1. Darboux Integrals	601
2. Order Properties of the Darboux Integral	618
3. Algebraic Properties of the Darboux Integral	625
4. The Riemann Integral	629
5. Primitives	638
6. Fundamental Theorem of the Calculus	649
7. The Substitution Formula for Definite Integrals	656
8. Integration by Parts	666
9. Integration by the Method of Partial Fractions	672
Chapter XIV	
The Riemann Integral II	681
1. Uniform Convergence and R -Integrals	681
2. Mean-Value Theorems for Integrals	689
3. Young's Inequality and Some of Its Applications	694
4. Integral Form of the Remainder in Taylor's Theorem	697
5. Sets of Measure Zero. The Cantor Set	699

Chapter XV	
Improper Integrals. Elliptic Integrals and Functions	710
1. Introduction. Definitions	710
2. Comparison Tests for Convergence of Improper Integrals	715
3. Absolute and Conditional Convergence of Improper Integrals	719
4. Integral Representation of the Gamma Function	726
5. The Beta Function	729
6. Evaluation of $\int_0^{+\infty} (\sin x)/x dx$	735
7. Integral Tests for Convergence of Series	738
8. Jacobian Elliptic Functions	741
9. Addition Formulas	747
10. The Uniqueness of the s , c , and d in Theorem 8.1	752
11. Extending the Definition of the Jacobi Elliptic Functions	755
12. Other Elliptic Functions and Integrals	759
Bibliography	764
Index	765

Preface

There are a great deal of books on introductory analysis in print today, many written by mathematicians of the first rank. The publication of another such book therefore warrants a defense. I have taught analysis for many years and have used a variety of texts during this time. These books were of excellent quality mathematically but did not satisfy the needs of the students I was teaching. They were written for mathematicians but not for those who were first aspiring to attain that status. The desire to fill this gap gave rise to the writing of this book.

This book is intended to serve as a text for an introductory course in analysis. Its readers will most likely be mathematics, science, or engineering majors undertaking the last quarter of their undergraduate education. The aim of a first course in analysis is to provide the student with a sound foundation for analysis, to familiarize him with the kind of careful thinking used in advanced mathematics, and to provide him with tools for further work in it. The typical student we are dealing with has completed a three-semester calculus course and possibly an introductory course in differential equations. He may even have been exposed to a semester or two of modern algebra. All this time his training has most likely been intuitive with heuristics taking the place of proof. This may have been appropriate for that stage of his development. However, once he enters the analysis course he is subject to an abrupt change in the point of view and finds that much more is demanded of him in the way of rigorous and sound deductive thinking. In writing the book we have this student in mind. It is intended to ease him into his next, more mature stage of mathematical development.

Throughout the text we adhere to the spirit of careful reasoning and rigor

that the course demands. We deal with the problem of student adjustment to the stricter standards of rigor demanded by slowing down the pace at which topics are covered and by providing much more detail in the proofs than is customary in most texts. Secondly, although the book contains its share of abstract and general results, it concentrates on the specific and concrete by applying these theorems to gain information about some of the important functions of analysis. Students are often presented and even have proved for them theorems of great theoretical significance without being given the opportunity of seeing them “in action” and applied in a non-trivial way. In our opinion, good pedagogy in mathematics should give substance to abstract and general results by demonstrating their power.

This book is concerned with real-valued functions of one real variable. There is a chapter on complex numbers, but these play a secondary role in the development of the material, since they are used mainly as computational aids to obtain results about trigonometric sums.

For pedagogical reasons we avoid “slick” proofs and sacrifice brevity for straightforwardness.

The material is developed deductively from axioms for the real numbers. The book is self-contained except for some theorems in finite sets (stated without proof in Chapter II) and the last theorem in Chapter XIV. In the main, any geometry that is included is there for purposes of visualization and illustration and is not part of the development. Very little is required from the reader in the way of background. However, we hope that he has the desire and ability to follow a deductive argument and is not afraid of elementary algebraic manipulation. In short, we would like the reader to possess some “mathematical maturity.” The book’s aim is to obtain all its results as logical consequences of the fifteen axioms for the real numbers listed in Chapter I.

The material is presented sequentially in “theorem–proof–theorem” fashion and is interspersed with definitions, examples, remarks, and problems. Even if the reader does not solve all the problems, we expect him to read each one and to understand the result contained in it. In many cases the results cited in the problems are used as proofs of later theorems and constitute part of the development. When the reader is asked, in a problem, to prove a result which is used later, this usually involves paralleling work already done in the text.

Chapters are denoted by Roman numerals and are separated into sections. Results are referred to by labeling them with the chapter, section, and the order in which they appear in the section. For example, Theorem X.6.2 refers to the second theorem of section 2 in Chapter X. When referring to a result in the same chapter, the Roman numeral indicating the chapter is omitted. Thus, in Chapter X, Theorem X.6.2 is referred to as Theorem 6.2.

We also mention a notational matter. The open interval with left endpoint a and right endpoint b is written in the book as $(a; b)$ using a

semi-colon between a and b , rather than as (a,b) . The latter symbol is reserved for the ordered pair consisting of a and b and we wish to avoid confusion.

I owe a special debt of gratitude to my friend and former colleague Professor Abe Shenitzer of York University in Ontario, Canada, for patiently reading through the manuscript and editing it for readability.

My son Joseph also deserves special thanks for reading most of the material, pointing out errors where he saw them, and making some valuable suggestions.

Thanks are due to Professors Eugene Levine and Ida Sussman, colleagues of mine at Adelphi University, and Professor Gerson Sparer of Pratt Institute, for reading different versions of the manuscript.

Ms. Maie Croner typed almost all of the manuscript. Her skill and accuracy made the task of readying it for publication almost easy.

I am grateful to the staff at Springer-Verlag for their conscientious and careful production of the book.

To my wife Sylvia I give thanks for her patience through all the years the book was in preparation. תושלב"ע

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