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Roger Howe
Eng Chye Tan

Non-Abelian Harmonic Analysis

Applications of $SL(2,\mathbb{R})$

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To our parents:

We offer this book to you

who deserve far more.

Preface

This book mainly discusses the representation theory of the special linear group $SL(2,\mathbb{R})$, and some applications of this theory. In fact the emphasis is on the applications; the working title of the book while it was being written was "Some Things You Can Do with $SL(2)$." Some of the applications are outside representation theory, and some are to representation theory itself. The topics outside representation theory are mostly ones of substantial classical importance (Fourier analysis, Laplace equation, Huyghens' principle, Ergodic theory), while the ones inside representation theory mostly concern themes that have been central to Harish-Chandra's development of harmonic analysis on semisimple groups (his restriction theorem, regularity theorem, character formulas, and asymptotic decay of matrix coefficients and temperedness). We hope this mix of topics appeals to nonspecialists in representation theory by illustrating (without an interminable prolegomena) how representation theory can offer new perspectives on familiar topics and by offering some insight into some important themes in representation theory itself. Especially, we hope this book popularizes Harish-Chandra's restriction formula, which, besides being basic to his work, is simply a beautiful example of Fourier analysis on Euclidean space. We also hope representation theorists will enjoy seeing examples of how their subject can be used and will be stimulated by some of the viewpoints offered on representation-theoretic issues.

Concentrating on $SL(2,\mathbb{R})$ reduces the amount of preparation and permits much more explicit computations than are possible for more general groups. Nevertheless, a fair amount of preliminary material is still needed and is reviewed in Chapters I and III, but this book is not self-contained. It is not an "Introduction to . . . "; it is a selection of topics, requiring various amounts of background. The needed background is mostly functional analysis, some measure theory, linear topological spaces, spectral theory, distribution theory, and a lot of patience for algebra. The basic facts about differentiable manifolds are sometimes used, and some acquaintance with Lie groups and Lie algebras will probably come in handy, although no intimacy is presumed. Though we hope this book conveys the flavor of the representation theory of $SL(2,\mathbb{R})$, it is not even a thorough treatment of that limited topic. It is to a large extent complementary to Serge Lang's text $SL_2(\mathbb{R})$ [1], which could be profitably read in addition to this book for a more rounded and systematic picture. We have attempted to keep our discussion concrete by illustrating phenomena valid on general groups by means of specific examples. Many of the scenes are set in Euclidean space.

Many exercises with extensive hints or guidance have been included to help bridge gaps in the discussion, as well as to point to supplementary topics. We hope that these devices will make the book accessible to second-year graduate students.

Chapter I gives a brief review of the background needed for the more particular discussion including basic definitions and facts about Lie groups, Lie algebras and their linear representations; Fourier transform and elements of distribution theory; and a formulation of spectral theory adapted to representations of $\mathbb{R}^n$, rather than a single self-adjoint operator. Chapter II discusses representations of the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$. Our basic technique is the concrete raising- and lowering-operator approach familiar from both the mathematical and physical literature. Nevertheless, we treat several topics, motivated specifically by the applications of Chapter IV, which are either hard to find or do not exist in the literature. It seems worth noting that the representations one needs to deal with the problems studied in Chapter IV are for the most part neither unitary nor irreducible: they are sums of indecomposable representations, and the indecomposability plays a key role in all the applications. Indeed, it is satisfying to observe how closely the combinatorics of submodule structure of the relevant representations of $\mathfrak{sl}(2)$ control (or, if you prefer, mimic) the associated analytic phenomena.

Chapter III is a brief study of the representations of $SL(2, \mathbb{R})$. We realize a family of representations as spaces of functions on $\mathbb{R}^2$. An important consequence of this is that of the irreducible representations of the $\mathfrak{sl}(2)$ construction in Chapter II, all those that could plausibly be derived from representations of the group $SL(2, \mathbb{R})$ are so derived. The unitarity of these representations is determined. This chapter also provides a brief introduction to the oscillator representation, which is the key to all the applications in Chapter IV.

Chapter IV applies the developments of the first three chapters to topics in harmonic analysis on $\mathbb{R}^n$. The Bochner periodicity relations for the Fourier transform of functions transforming in a given way under the orthogonal group, in a particularly elegant formulation due to R. Coifman; Harish-Chandra's restriction formula for the unitary group $U(n)$; fundamental solutions for the definite and indefinite Laplacians; Huyghens' principle for the wave equation; Harish-Chandra's regularity theorem and Rossmann-Harish-Chandra-Kirillov character formula (for the case of $\mathfrak{sl}(2)$ only). The constant refrain behind all these melodies is precise control over the $SL(2, \mathbb{R})$-module structure (via the oscillator representation) of distributions invariant, or almost invariant, under some indefinite orthogonal group. Even in the very classical and by now very elementary case of the usual, definite Laplacian, we feel that representation theory adds something to the picture.

Chapter V treats another set of topics, connected with the behavior of matrix coefficients. Matrix coefficients are the glue that binds representation theory to many aspects of classical analysis, especially the theory of special functions, and they have also played a key role in classifying representations of semisimple groups. Our focus in this chapter is rather limited: we prove several estimates, one qualitative and another more quantitative, concerning the decay at infinity of matrix coefficients on the group. These estimates have several applications in, for example, the structure of discrete groups, ergodic theory, and estimation of Hecke eigenvalues and Fourier coefficients of automorphic forms. The book concludes with a description of some of these applications.

Recently, several books have been published that develop some or most of the theory of representations of semisimple groups. We mention particularly the books of Knapp [1] and Wallach [1]. This book is not subsumed in either of those and is quite different in intent. Where those books present thorough surveys of broad domains, this offers a day hike to a nearby waterfall. If those books are like *War and Peace*, this one is more like *The Hunting Sketches*. Nevertheless, there is certainly a connection. Anyone who masters those books will find this one quite easy. Conversely, we hope this book will make some parts of those others easier to understand. This book might fit in the middle of a program to learn semisimple harmonic analysis, preceded by Lang's $SL_2(\mathbb{R})$, followed by Knapp [1] or Wallach [1]. We should also mention that Wallach's remarkable work has had substantial impact on this volume; in particular, our Chapter IV is in part a meditation on Appendix 5 of his Chapter 7. Late in the production of this book we became aware of the book "An Introduction to Harmonic Analysis on Semisimple Lie Groups" by V. S. Varadarajan. It provides a treatment of some of the same topics as this book, from a point of view much more faithful to Harish-Chandra's original treatment, and we recommend it for this feature.

This book is based on a course given by the first author at Yale University in Spring 1989. Notes taken by class members were organized and considerably amplified and supplemented by the second author, then reworked several times jointly. It is a pleasure to thank the Mathematics Department of Yale University for its beneficial atmosphere and the Mathematics Departments of Rutgers University and the National University of Singapore for their support in 1989–1990. We appreciate the encouragement given us by Ms. Ulricke Schmickler-Hirzebruch of Springer-Verlag from an early stage in the project. We warmly thank Chun-Chung Hsieh, Andrea Nahmod, Beatrice Polloni, Ze'ev Rudnick, Xiao-Xi Xue, Sijue Wu, and Chen-Bo Zhu for preparing notes from which this book evolved. We are also grateful to Helmer Aslaksen, Chuan Chong Chen, Shih Ping Chan, and Hwee Huat Tan for their technical advice as regards the AMS-TEX, LATEX and AMS-LATEX softwares. We appreciate a careful reading by Tôru Umeda and Tomasz Przebinda's efforts to eliminate obscurities. Very special thanks is due Lo-Mun Ng for her consistent support of the second author and in particular for most of the typing.

Notations

For convenience, we use the following convention :

$$A^{\pm} = B \pm C^{\pm}$$

(where A, B, and C are expressions or symbols) to describe the following relations:

$$\text{(a)} \quad A^{+} = B + C^{+} \quad \text{and} \quad \text{(b)} \quad A^{-} = B - C^{-}.$$

Unless otherwise stated, we adopt the following notations:

$\{x \in A \mid p,\ q\}$	the set of x in A such that p and q hold
$A - B$	set theoretic complement of B in A
A^n	n-tuples $(a_1, \dots, a_n)$ where $a_j \in A,\ j = 1, \dots, n$
$\emptyset$	empty set
$\mathbb{R}$	{ real numbers }
$\mathbb{R}_+$	{ real numbers ≥ 0}
$\mathbb{R}^\times$	{ nonzero real numbers }
$\mathbb{R}_+^\times$	{ nonzero positive real numbers }
$\mathbb{C}$	{ complex numbers }
$\mathbb{C}^\times$	{ nonzero complex numbers }
$\mathbb{Z}$	{ integers }
$\mathbb{Z}_+$	{ integers ≥ 0}
$\mathbb{Z}_-$	{ integers ≤ 0}
i	$\sqrt{-1}$
V^*	complex dual of vector space V
End V	maps of vector space V into itself
Hom (V, W)	maps from V to W
I or 1	identity (transformation)
I_n	n by n identity matrix
ker T	kernel of the operator T
Im T	image of the operator T

$E_{j,k}$	standard matrix unit, with 1 on the (j,k) entry and zero everywhere else
$\overline{S}$	closure of S or complex conjugate of S (should be understood from context)
$S^{\perp}$	orthogonal complement of S in a Hilbert space
A^t	transpose of a matrix A
$\sum$	direct sum (orthogonal direct sum in the context of Hilbert spaces)
$+$	direct sum
$\langle x,t\rangle$ or $x \cdot t$	dot product of two vectors x and t in $\mathbb{R}^n$ or Hermitian dot product of two vectors x and t in $\mathbb{C}^n$
$f\|_E$	restriction of a function f to E

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