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Preface

This book has grown out of lectures and courses in calculus of variations and optimization taught for many years at the University of Michigan to graduate students at various stages of their careers, and always to a mixed audience of students in mathematics and engineering. It attempts to present a balanced view of the subject, giving some emphasis to its connections with the classical theory and to a number of those problems of economics and engineering which have motivated so many of the present developments, as well as presenting aspects of the current theory, particularly value theory and existence theorems. However, the presentation of the theory is connected to and accompanied by many concrete problems of optimization, classical and modern, some more technical and some less so, some discussed in detail and some only sketched or proposed as exercises.

No single part of the subject (such as the existence theorems, or the more traditional approach based on necessary conditions and on sufficient conditions, or the more recent one based on value function theory) can give a sufficient representation of the whole subject. This holds particularly for the existence theorems, some of which have been conceived to apply to certain large classes of problems of optimization.

For all these reasons it is essential to present many examples (Chapters 3 and 6) before the existence theorems (Chapters 9 and 11–16), and to investigate these examples by means of the usual necessary conditions, sufficient conditions, and value function theory.

This book only considers nonparametric problems of the calculus of variations in one independent variable and problems of optimal control monitored by ordinary differential equations. Multidimensional problems monitored by partial differential equations, parametric problems with simple and multiple integrals, parametric problems of optimal control, and related questions of nonlinear integration will be presented elsewhere.

Chapter 1 is introductory. The many types of problems of optimization are reviewed and their intricate relationships illustrated.

Chapter 2 presents the necessary conditions, the sufficient conditions, and the value function theory for classical problems of the calculus of variations. In particular, the Weierstrass necessary condition is being studied as a necessary condition for lower semicontinuity on a given trajectory.

Chapter 3 consists mainly of examples. In particular, it includes points of Ramsey's theory of economic growth, and points of theoretical mechanics.

Chapters 4 and 5 deal with problems of optimal control. They contain a statement of the necessary condition, a detailed discussion of the transversality relation in its generality, a discussion of Bellman's value function theory, and a statement of Boltyanskii's sufficient condition in terms of regular synthesis.

Chapter 6 consists mainly of examples. In particular, points of the neoclassical theory of economic growth are also studied.

Chapter 7 presents two proofs of the necessary condition for problems of optimal control.

Chapter 8 contains preparatory material for existence theorems, in particular, Kuratowski's and Ryll-Nardzewski's selection theorems, McShane's and Warfield's implicit function theorem, and some simple forms of the lower closure theorem for uniform convergence.

Chapter 9 deals with existence theorems for problems of optimal control with continuous data and compact control space. These are essentially Filippov's existence theorems. The proofs in this chapter are designed to be elementary in the sense that mere uniform convergence is involved, whereas in Chapters 10 and 11 use is made of weak convergence in L_1 .

Chapter 10 presents the Banach-Saks-Mazur theorem, the Dunford-Pettis theorem, and closure, lower closure, and lower semicontinuity theorems for weak convergence in L_1 .

Chapter 11 deals with existence theorems based on weak convergence. Existence theorems are proved for Lagrange problems with an integrand which is an extended function, and then existence theorems are derived for problems of optimal control. Moreover, existence theorems are proved for problems with comparison functionals, for isoperimetric problems, and specifically for problems which are linear in the derivatives, or in the controls. In particular, this chapter contains a present day version of the theorem established by Tonelli in 1914 for problems with a uniform growth property.

In Chapter 12 existence theorems are presented where a growth assumption fails at the points of a "slender" set. In Chapter 13 existence theorems under numerous analytical conditions are studied. Chapter 14 deals with existence theorems for problems without growth assumptions. Chapter 15 presents theorems based on mere pointwise convergence. Chapter 16 deals with Neustadt-type existence theorems for problems with no convexity assumptions.

Chapter 17 covers a few points of convex analysis including duality, and the equivalence of a certain concept of upper semicontinuity for sets with

the concept of seminormality of Tonelli and McShane for functions, and suitable properties in terms of convex analysis.

Chapter 18 covers questions of approximation of usual and generalized trajectories.

Each chapter contains examples and exercises. Bibliographical notes at the end of each chapter provide some historical background and direct the reader to the literature in the field.

A number of parts in this book are in smaller print so as to facilitate, at a first reading, a faster perusal. The small-print passages include most of the examples and remarks, several of the complementary considerations, and a number of the more technical proofs.

I wish to thank the many associates and graduate students who, with their remarks and suggestions upon reading these notes, have contributed so much to make this presentation a reality.

Finally, I wish to express my appreciation to Springer-Verlag for their accomplished handling of the manuscript, their understanding and patience.

Contents

Chapter 1	
Problems of Optimization—A General View	1
1.1 Classical Lagrange Problems of the Calculus of Variations	1
1.2 Classical Lagrange Problems with Constraints on the Derivatives	3
1.3 Classical Bolza Problems of the Calculus of Variations	4
1.4 Classical Problems Depending on Derivatives of Higher Order	5
1.5 Examples of Classical Problems of the Calculus of Variations	5
1.6 Remarks	8
1.7 The Mayer Problems of Optimal Control	9
1.8 Lagrange and Bolza Problems of Optimal Control	11
1.9 Theoretical Equivalence of Mayer, Lagrange, and Bolza Problems of Optimal Control. Problems of the Calculus of Variations as Problems of Optimal Control	11
1.10 Examples of Problems of Optimal Control	14
1.11 Exercises	14
1.12 The Mayer Problems in Terms of Orienter Fields	15
1.13 The Lagrange Problems of Control as Problems of the Calculus of Variations with Constraints on the Derivatives	16
1.14 Generalized Solutions	18
Bibliographical Notes	23
Chapter 2	
The Classical Problems of the Calculus of Variations: Necessary Conditions and Sufficient Conditions; Convexity and Lower Semicontinuity	24
2.1 Minima and Maxima for Lagrange Problems of the Calculus of Variations	24
2.2 Statement of Necessary Conditions	30
2.3 Necessary Conditions in Terms of Gateau Derivatives	37
	ix

2.4	Proofs of the Necessary Conditions and of Their Invariant Character	42
2.5	Jacobi's Necessary Condition	53
2.6	Smoothness Properties of Optimal Solutions	57
2.7	Proof of the Euler and DuBois-Reymond Conditions in the Unbounded Case	61
2.8	Proof of the Transversality Relations	64
2.9	The String Property and a Form of Jacobi's Necessary Condition	65
2.10	An Elementary Proof of Weierstrass's Necessary Condition	69
2.11	Classical Fields and Weierstrass's Sufficient Conditions	70
2.12	More Sufficient Conditions	83
2.13	Value Function and Further Sufficient Conditions	89
2.14	Uniform Convergence and Other Modes of Convergence	98
2.15	Semicontinuity of Functionals	100
2.16	Remarks on Convex Sets and Convex Real Valued Functions	101
2.17	A Lemma Concerning Convex Integrands	102
2.18	Convexity and Lower Semicontinuity: A Necessary and Sufficient Condition	103
2.19	Convexity as a Necessary Condition for Lower Semicontinuity	104
2.20	Statement of an Existence Theorem for Lagrange Problems of the Calculus of Variations	111
	Bibliographical Notes	114
Chapter 3		
	Examples and Exercises on Classical Problems	116
3.1	An Introductory Example	116
3.2	Geodesics	117
3.3	Exercises	120
3.4	Fermat's Principle	120
3.5	The Ramsay Model of Economic Growth	123
3.6	Two Isoperimetric Problems	125
3.7	More Examples of Classical Problems	127
3.8	Miscellaneous Exercises	131
3.9	The Integral $I = \int (x'^2 - x^2) dt$	132
3.10	The Integral $I = \int x x'^2 dt$	135
3.11	The Integral $I = \int x'^2 (1 + x')^2 dt$	136
3.12	Brachistochrone, or Path of Quickest Descent	139
3.13	Surface of Revolution of Minimum Area	143
3.14	The Principles of Mechanics	149
	Bibliographical Notes	158
Chapter 4		
	Statement of the Necessary Condition for Mayer Problems of Optimal Control	159
4.1	Some General Assumptions	159
4.2	The Necessary Condition for Mayer Problems of Optimal Control	162
4.3	Statement of an Existence Theorem for Mayer's Problems of Optimal Control	173
4.4	Examples of Transversality Relations for Mayer Problems	174
4.5	The Value Function	181
4.6	Sufficient Conditions	184

4.7 Appendix: Derivation of Some of the Classical Necessary Conditions of Section 2.1 from the Necessary Condition for Mayer Problems of Optimal Control	189
4.8 Appendix: Derivation of the Classical Necessary Condition for Isoperimetric Problems from the Necessary Condition for Mayer Problems of Optimal Control	191
4.9 Appendix: Derivation of the Classical Necessary Condition for Lagrange Problems of the Calculus of Variations with Differential Equations as Constraints	193
Bibliographical Notes	195
Chapter 5 Lagrange and Bolza Problems of Optimal Control and Other Problems	196
5.1 The Necessary Condition for Bolza and Lagrange Problems of Optimal Control	196
5.2 Derivation of Properties (P1')—(P4') from (P1)—(P4)	199
5.3 Examples of Applications of the Necessary Conditions for Lagrange Problems of Optimal Control	201
5.4 The Value Function	202
5.5 Sufficient Conditions for the Bolza Problem	204
Bibliographical Notes	205
Chapter 6 Examples and Exercises on Optimal Control	206
6.1 Stabilization of a Material Point Moving on a Straight Line under a Limited External Force	206
6.2 Stabilization of a Material Point under an Elastic Force and a Limited External Force	209
6.3 Minimum Time Stabilization of a Reentry Vehicle	213
6.4 Soft Landing on the Moon	214
6.5 Three More Problems on the Stabilization of a Point Moving on a Straight Line	217
6.6 Exercises	218
6.7 Optimal Economic Growth	221
6.8 Two More Classical Problems	224
6.9 The Navigation Problem	227
Bibliographical Notes	232
Chapter 7 Proofs of the Necessary Condition for Control Problems and Related Topics	233
7.1 Description of the Problem of Optimization	233
7.2 Sketch of the Proofs	235
7.3 The First Proof	236
7.4 Second Proof of the Necessary Condition	256
7.5 Proof of Boltyanskii's Statements (4.6.iv–v)	264
Bibliographical Notes	269

Chapter 8	
The Implicit Function Theorem and the Elementary Closure Theorem	271
8.1 Remarks on Semicontinuous Functionals	271
8.2 The Implicit Function Theorem	275
8.3 Selection Theorems	280
8.4 Convexity, Carathéodory's Theorem, Extreme Points	286
8.5 Upper Semicontinuity Properties of Set Valued Functions	290
8.6 The Elementary Closure Theorem	298
8.7 Some Fatou-Like Lemmas	301
8.8 Lower Closure Theorems with Respect to Uniform Convergence	302
Bibliographical Notes	307
Chapter 9	
Existence Theorems: The Bounded, or Elementary, Case	309
9.1 Ascoli's Theorem	309
9.2 Filippov's Existence Theorem for Mayer Problems of Optimal Control	310
9.3 Filippov's Existence Theorem for Lagrange and Bolza Problems of Optimal Control	313
9.4 Elimination of the Hypothesis that A Is Compact in Filippov's Theorem for Mayer Problems	317
9.5 Elimination of the Hypothesis that A Is Compact in Filippov's Theorem for Lagrange and Bolza Problems	318
9.6 Examples	319
Bibliographical Notes	324
Chapter 10	
Closure and Lower Closure Theorems under Weak Convergence	325
10.1 The Banach–Saks–Mazur Theorem	325
10.2 Absolute Integrability and Related Concepts	326
10.3 An Equivalence Theorem	329
10.4 A Few Remarks on Growth Conditions	330
10.5 The Growth Property (ϕ) Implies Property (Q)	333
10.6 Closure Theorems for Orientor Fields Based on Weak Convergence	340
10.7 Lower Closure Theorems for Orientor Fields Based on Weak Convergence	342
10.8 Lower Semicontinuity in the Topology of Weak Convergence	350
10.9 Necessary and Sufficient Conditions for Lower Closure	359
Bibliographical Notes	364
Chapter 11	
Existence Theorems: Weak Convergence and Growth Conditions	367
11.1 Existence Theorems for Orientor Fields and Extended Problems	367
11.2 Elimination of the Hypothesis that A Is Bounded in Theorems (11.1. i–iv)	379
11.3 Examples	381
11.4 Existence Theorems for Problems of Optimal Control with Unbounded Strategies	383

11.5 Elimination of the Hypothesis that A Is Bounded in Theorems (11.4.i–v)	396
11.6 Examples	397
11.7 Counterexamples	398
Bibliographical Notes	399
Chapter 12	
Existence Theorems: The Case of an Exceptional Set of No Growth	403
12.1 The Case of No Growth at the Points of a Slender Set. Lower Closure Theorems.	403
12.2 Existence Theorems for Extended Free Problems with an Exceptional Slender Set	411
12.3 Existence Theorems for Problems of Optimal Control with an Exceptional Slender Set	413
12.4 Examples	414
12.5 Counterexamples	415
Bibliographical Notes	415
Chapter 13	
Existence Theorems: The Use of Lipschitz and Tempered Growth Conditions	417
13.1 An Existence Theorem under Condition (D)	417
13.2 Conditions of the F, G, and H Types Each Implying Property (D) and Weak Property (Q)	422
13.3 Examples	427
Bibliographical Notes	429
Chapter 14	
Existence Theorems: Problems of Slow Growth	430
14.1 Parametric Curves and Integrals	430
14.2 Transformation of Nonparametric into Parametric Integrals	436
14.3 Existence Theorems for (Nonparametric) Problems of Slow Growth	438
14.4 Examples	440
Bibliographical Notes	442
Chapter 15	
Existence Theorems: The Use of Mere Pointwise Convergence on the Trajectories	443
15.1 The Helly Theorem	443
15.2 Closure Theorems with Components Converging Only Pointwise	444
15.3 Existence Theorems for Extended Problems Based on Pointwise Convergence	446
15.4 Existence Theorems for Problems of Optimal Control Based on Pointwise Convergence	450
15.5 Exercises	451
Bibliographical Notes	452

Chapter 16	
Existence Theorems: Problems with No Convexity Assumptions	453
16.1 Lyapunov Type Theorems	453
16.2 The Neustadt Theorem for Mayer Problems with Bounded Controls	458
16.3 The Bang-Bang Theorem	460
16.4 The Neustadt Theorem for Lagrange and Bolza Problems with Bounded Controls	462
16.5 The Case of Unbounded Controls	464
16.6 Examples for the Unbounded Case	471
16.7 Problems of the Calculus of Variations without Convexity Assumptions	472
Bibliographical Notes	473
Chapter 17	
Duality and Upper Semicontinuity of Set Valued Functions	474
17.1 Convex Functions on a Set	474
17.2 The Function $T(x; z)$	478
17.3 Seminormality	481
17.4 Criteria for Property (Q)	482
17.5 A Characterization of Property (Q) for the Sets $\tilde{Q}(t, x)$ in Terms of Seminormality	486
17.6 Duality and Another Characterization of Property (Q) in Terms of Duality	488
17.7 Characterization of Optimal Solutions in Terms of Duality	496
17.8 Property (Q) as an Extension of Maximal Monotonicity	500
Bibliographical Notes	502
Chapter 18	
Approximation of Usual and of Generalized Solutions	503
18.1 The Gronwall Lemma	503
18.2 Approximation of AC Solutions by Means of C^1 Solutions	504
18.3 The Brouwer Fixed Point Theorem	508
18.4 Further Results Concerning the Approximation of AC Trajectories by Means of C^1 Trajectories	508
18.5 The Infimum for AC Solutions Can Be Lower than the One for C^1 Solutions	514
18.6 Approximation of Generalized Solutions by Means of Usual Solutions	517
18.7 The Infimum for Generalized Solutions Can Be Lower than the One for Usual Solutions	519
Bibliographical Notes	520
Bibliography	523
Author Index	537
Subject Index	540

To Isotta, always