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Theory of Operator Algebras I

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Introduction

Mathematics for infinite dimensional objects is becoming more and more important today both in theory and application. *Rings of operators*, renamed *von Neumann algebras* by J. Dixmier, were first introduced by J. von Neumann fifty years ago, 1929, in [254] with his grand aim of giving a sound foundation to mathematical sciences of infinite nature. J. von Neumann and his collaborator F. J. Murray laid down the foundation for this new field of mathematics, *operator algebras*, in a series of papers, [240], [241], [242], [257] and [259], during the period of the 1930s and early in the 1940s. In the introduction to this series of investigations, they stated *Their solution* (to the problems of understanding rings of operators)[1] *seems to be essential for the further advance of abstract operator theory in Hilbert space under several aspects. First, the formal calculus with operator-rings leads to them. Second, our attempts to generalize the theory of unitary group-representations essentially beyond their classical frame have always been blocked by the unsolved questions connected with these problems. Third, various aspects of the quantum mechanical formalism suggest strongly the elucidation of this subject. Fourth, the knowledge obtained in these investigations gives an approach to a class of abstract algebras without a finite basis, which seems to differ essentially from all types hitherto investigated.* Since then there has appeared a large volume of literature, and a great deal of progress has been achieved by many mathematicians. The motivations of Murray and von Neumann seem to have been fully verified. Many important results and powerful techniques were added to the theory. Various related fields of mathematics have emerged, and a number of topics in this subject have branched out to independent fields.

[1] Added by the author.

The main characteristic of this subject can be stated as a complex of analysis and algebra: the results are phrased in algebraic terms, while the techniques are highly analytic. Sometimes, one might run into problems directly related to the foundation of mathematics such as the continuum hypothesis. One might be amazed to realize the possibility of such an elaborated algebraic structure in this wild area involving high degrees of infinity.

The theory of operator algebras is concerned with self-adjoint algebras of bounded linear operators on a Hilbert space closed under the norm topology, C^*-algebras, or the weak operator topology, von Neumann algebras. C^*-algebras are characterized as a special class of Banach algebras by means of a simple system of axioms. A concrete realization of a C^*-algebra as an algebra of operators on a Hilbert space is regarded as a representation of the algebra. Thus, the study of C^*-algebras consists of two parts: one is concerned with the intrinsic structure of algebras and the other deals with the representations of a C^*-algebra. Needless to say, these two parts are closely related, and indeed the algebraic structure of a C^*-algebra is studied through various representations of the algebra. Thus, this division of the theory stays at a formal level. Nevertheless, the separation of problems has positive effects: for instance, a systematic usage of inequivalent representations of a C^*-algebra provides flexible techniques even if it is given as a concrete algebra of operators on a specially chosen Hilbert space. Indeed, this freedom in choosing an appropriate representation is one of the main merits of the axiomatic approach to operator algebras.

Being infinite dimensional, our problems require careful investigation of approximation process; thus the study of topological structures is inevitable. For this reason, the topological, analytical aspect of operator algebras receives more of our attention than the algebraic aspect in this first volume. After establishing the basic foundation in Chapter I, the Banach space duality for operator algebras will be studied throughout the text. The reader will find a strong similarity between our theory and measure theory on locally compact spaces. In fact, the study of abelian C^*-algebras will be reduced to that of locally compact spaces, and a substantial part of our theory is called noncommutative integration theory.

Each chapter begins with an introduction to its basic facts. Sections and paragraphs with * sign are somewhat technical; the reader who wants to get rather a quick grasp of the theory may postpone these parts. The sign ** indicates the end of the technical paragraph. Comments and historic background are placed at the end of each chapter and some sections as notes. Complements to a section or a chapter and some results of special interest are stated as exercises with † sign and references.

In the succeeding volume, the author will discuss further, among other topics, noncommutative integration theory, the so-called Tomita–Takesaki theory, automorphism groups of operator algebras, crossed products, infinite tensor products, the structure of von Neumann algebras of type III, approximately finite dimensional von Neumann algebras, and the existence of a continuum of nonisomorphic factors.

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