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Preface to the Second Edition

In preparing the second edition, I have taken advantage of the opportunity to correct errors as well as revise the presentation in many places. New material has been included, in addition, reflecting relevant recent work.

The help of many colleagues (and especially Professor J. Stoer) in ferreting out errors is gratefully acknowledged. I also owe special thanks to Professor V. Sazonov for many discussions on the white noise theory in Chapter 6.

February, 1981

A. V. BALAKRISHNAN

Preface to the First Edition

The title "Applied Functional Analysis" is intended to be short for "Functional analysis in a Hilbert space and certain of its applications," the applications being drawn mostly from areas variously referred to as system optimization or control systems or systems analysis.

One of the signs of the times is a discernible tilt toward application in mathematics and conversely a greater level of mathematical sophistication in the application areas such as economics or system science, both spurred undoubtedly by the heightening pace of digital computer usage. This book is an entry into this twilight zone. The aspects of functional analysis treated here are rapidly becoming essential in the training at the advance graduate level of system scientists and/or mathematical economists. There are of course now available many excellent treatises on functional analysis. However, the very fact of the comprehensive coverage makes it difficult of access to the application-minded user. Also, the high degree of generality, the watermark of mathematical achievement, is often at the expense of the richer results obtainable in the more highly structured cases common in applications. It is with some of these thoughts in mind that I have dealt exclusively with analysis in a Hilbert space and emphasized such special topics as Volterra operators and Hilbert–Schmidt operators; dissipative compact semigroups; and factorization theorems for positive definite operators, to name a few. Many topics in functional analysis *per se* have had to be totally shelved or otherwise abridged considerably mostly based on considerations of significance in application, but also to keep the size of the volume within reasonable bounds.

Another point is that the abstract theory is sometimes easier than the applications. This is true for instance in the case of semigroup theory where the generation theorems, for example, are far easier than showing that a particular partial differential equation generates a semigroup. Indeed a novice is bewildered by a seemingly endless variety of approaches to boundary value problems, even to the notion of what is meant by boundary value. Here I have taken some pains to illustrate by examples how the abstract theory relates to problems in partial differential equations without of course any claim to completeness.

Of the six chapters in the book, three deal specifically with applications topics. These are Chapter 2 on convex sets and convex programming in a Hilbert space; Chapter 5 on deterministic control problems and Chapter 6 on stochastic optimization problems. Chapter 6 is unusual in that it exploits the theory of finitely additive probability measures on a Hilbert space (in contrast to the more standard Wiener measure on the space of continuous functions). This chapter also contains some original material.

The remaining chapters (about two thirds of the book) are devoted to functional analysis and semigroups within a Hilbert space framework. The basic properties of Hilbert spaces and some of the fundamental theorems central to what follows are in the beginning chapter. The background so built up is sufficient to consider applications to convex programming problems in the second chapter. It is possible to proceed directly from Chapter 1 to Chapter 3 featuring the theory of linear operators in a Hilbert space. L_2-distributional derivatives are studied as examples of unbounded operators and associated notion of Sobolev spaces. Operators over separable Hilbert spaces receive special attention as well as L_2 spaces over Hilbert spaces. The final section in Chapter 3 is devoted to nonlinear operators, or more accurately, polynomials and analytic functions. We go on to semigroup theory in Chapter 5, again emphasizing the more specialized cases such as compact semigroups and Hilbert–Schmidt semigroups. Semigroup theory in a Hilbert space strikes the right balance for our purposes between the too general and the too particular; for example, it provides a general enough framework for optimization problems involving partial differential equations without getting lost in the details of the particular equations. The concepts of controllability and observability important in system theory are examined in the semigroup theoretic setting. An example illustrates the application to nonhomogeneous boundary value problems. A final section deals with a special class of evolution equations that arise as perturbations of the semigroup equation.

The book is a revised and enlarged version of the author's *Introduction to Optimization Theory in a Hilbert Space*, No. 42 in the Springer-Verlag Lecture Series on Economics and Mathematical Systems Theory, and has been used in graduate courses given in the Department of Mathematics and the Department of System Science. The prerequisites are the standard graduate courses in real and complex variables and concommitant material such as Fourier transforms; material on function spaces usually included in real analysis texts would be helpful background since the bare definitions given here in introductory sections may be inadequate for a firm grasp.

Similarly in the applications chapters, some familiarity with control problems in finite dimensions would be helpful.

Many students, past and present, have helped in improving the presentation: D. Washburn, Claude Benchimol and Frank Tung, in particular. Dr. J. Mersky helped with proofreading. Dr. J. Ruzicka rendered much needed assistance throughout the various stages of the manuscript.

I am indebted to Regina Safdie for her endurance typing and to W. Kaufmann-Bühler for interest and encouragement. Grateful acknowledgement is made of the financial support in part under a research grant from the Applied Mathematics Division, AFOSR-USAF, monitored by Colonel W. Rabe.

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