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Colin Sparrow

The Lorenz Equations: Bifurcations, Chaos, and Strange Attractors

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Preface

The equations which we are going to study in these notes were first presented in 1963 by E. N. Lorenz. They define a three-dimensional system of ordinary differential equations that depends on three real positive parameters. As we vary the parameters, we change the behaviour of the flow determined by the equations. For some parameter values, numerically computed solutions of the equations oscillate, apparently forever, in the pseudo-random way we now call "chaotic"; this is the main reason for the immense amount of interest generated by the equations in the eighteen years since Lorenz first presented them. In addition, there are some parameter values for which we see "preturbulence", a phenomenon in which trajectories oscillate chaotically for long periods of time before finally settling down to stable stationary or stable periodic behaviour, others in which we see "intermittent chaos", where trajectories alternate between chaotic and apparently stable periodic behaviours, and yet others in which we see "noisy periodicity", where trajectories appear chaotic though they stay very close to a non-stable periodic orbit.

Though the Lorenz equations were not much studied in the years between 1963 and 1975, the number of man, woman, and computer hours spent on them in recent years - since they came to the general attention of mathematicians and other researchers - must be truly immense. Besides this mainstream of "Lorenz" research, countless authors have quoted the equations as an example or referred to them in the course of some debate or other. Despite all this interest, there has never been an attempt to tie together all the various different parameter ranges. The first purpose of these notes is to attempt to fill that gap.

For some parts of these notes I can claim no great originality. I have, wherever possible, referenced other authors, but should warn that these notes are not intended to be a complete review of the literature. I have been guided by what I have seen, and apologize to those authors whom I have omitted or unintentionally misinterpreted. As often, the most notable omissions are likely to be Russian authors. Nowhere should my references be taken to indicate any firm belief that the author references wrote either the first, or the best, paper on a subject. In attempting to give as thorough a description as possible of the various bifurcations which lead from one well studied parameter range to another, I hope that I have added to, rather than subtracted from, the various separate contributions so far.

These notes are also intended to illustrate an approach. It cannot be claimed that the Lorenz equations show all the different behaviours of a general set of chaotic ordinary differential equations. Indeed, they possess certain special properties (such as a symmetry) that indicate that this cannot be so. Nonetheless, many other systems behave in ways which seem to be very similar to one or more of the behaviours shown by the Lorenz equations, and an understanding of the Lorenz equations can be expected to increase our understanding of these other systems. The way in which we obtain this understanding is to move constantly back and forth between theory, models which demonstrate those properties which we deem to be important at any particular time, and numerical experiments on the equations themselves. Thus, we avoid some of the pitfalls of more single-minded approaches. Those who seek to know, with mathematical certainty, what the Lorenz equations "do" will be disappointed. Most often we proceed only to the point where we know "beyond all reasonable doubt", and readers should always bear in mind that much of these notes is dependent on computer generated numerical output which can always be misleading for reasons outside our control or even outside our comprehension.

It is my hope that these notes will be comprehensible to those readers with no previous knowledge of the Lorenz equations (or any other chaotic differential equations), as well as being informative and interesting to the "experts". It is assumed that readers know a little about differential equations and the various simpler bifurcations which can occur as parameters change.

Readers may notice that there is very little discussion of the problem of "real world" turbulence. This is a deliberate policy. It is not that I believe that the study of chaotic ordinary differential equations can never be helpful in understanding real world phenomena; rather, I believe that until we know more about the behaviour of the finite-dimensional approximations that model the partial differential equations that model the world, I do not think I have anything very useful to say on the subject.

These notes are divided into nine chapters and eleven appendices. The first four chapters review what is known about the Lorenz equations in the most widely studied parameter ranges. Chapter 1 contains some general remarks and a description of those simple properties of the equations that can be deduced mathematically. In Chapter 2, we study the bifurcation associated with a homoclinic orbit. This study is more general than usual, since we shall see that there are many important homoclinic bifurcations in the Lorenz system. In Chapter 3, we describe the parameter range where it is believed that we have a well understood strange attractor in a whole interval of parameter values. Chapter 4 contains a description of the results of some simple numerical experiments in a parameter range where period doubling is observed. In Chapter 5, we attempt to reconcile Chapters 3 and 4. Using a combination of different numerical techniques and a careful theoretical analysis of the changes in the behaviour of the unstable manifold of the origin (which is dependent on our general knowledge of homoclinic bifurcations), we show how the behaviour changes from strange attractor to period doublings. In the process, we uncover various new properties of the Lorenz equations, including indications that our simple numerical experiments from Chapter 4 are misleading in various ways. Chapter 5 is probably not comprehensible without at least a quick reading of Chapters 1 through 4.

Chapter 6 is an attempt to justify the methods we have used earlier in the text to describe periodic orbits and other trajectories using sequences of symbols. In the process of this justification, we study the behaviour of the stable manifolds of the stationary points other than the origin. Chapter 6 could be omitted.

Chapter 7 contains an outline of an analysis of the behaviour when one of the parameters becomes large. This analysis goes considerably beyond earlier analyses and suggests that the "large r" behaviour may be qualitatively more complicated when the other two parameters are allowed to take values other than the usual ones. Chapter 7 could be read in isolation.

Chapter 8 contains a study of the Lorenz equations for parameter values suggested by Chapter 7. The expectations of qualitatively more complicated behaviour are confirmed. Nonetheless, the general theory and approach of Chapters 1 through 7 shows us that we can, with very few numerical experiments, make a large number of interesting statements about this more complicated behaviour. In particular, we discover a kind of bifurcation not previously observed in the Lorenz equations, and can explain exactly how this bifurcation fits into the more general picture.

Chapter 9 contains a brief summary, a quick examination of some of the approaches used by other authors on the Lorenz equations, and a brief discussion of some Lorenz-like equations at present under investigation.

Throughout Chapters 1-9, I have attempted to confine my attention to those things with direct relevance to the actual Lorenz equations (as opposed to models or simplifications of the equations). The Appendices are of several types. Some contain little bits of mathematics which, though of direct relevance to the discussion in the main body of the text, have been relegated to an appendix so as to avoid breaking the flow of the description; often we can proceed just by quoting the results from the relevant appendix. Some appendices are self-contained and describe results with application to more than the Lorenz equations. Examples are the appendices on homoclinic bifurcations, and on numerical techniques for the location and following (with changing parameter) of non-stable periodic orbits. These appendices may be of separate use to some readers; in any case it is convenient to have these results in one place since they are referred to several times in the main body of the text. One appendix contains a review of work on a geometric model of the Lorenz equations in a parameter range where the strange attractor is believed to exist.

The numerical integration of differential equations, on which these notes depend, was done using standard integrating packages. Most of the simulations have been done on two different machines, using different packages. These were variously based on Merson's method, a variable order Adam's method, and a variable order Runge-Kutta method, all of which produced similar results.

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Berkeley, California
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I have had the opportunity to talk to many mathematicians and others about the Lorenz equations; to all I say thank you. Bob Williams deserves a special mention, not only did he talk with me at great length in the early stages of my research, but he also had me to stay in his house in Evanston for a week. Various people have looked through different versions of these notes; I am grateful to Jack Hale and John Guckenheimer for some general remarks, and to Peter Swinnerton-Dyer, Andrew Fowler, Edgar Knobloch and Sian Stumbles for detailed criticisms. I would also like to thank Kate MacDougall who did a wonderful job of typing the final version, Eleanor Addison who drew all the figures that were not drawn by the computer, David Abrahamson who gave editorial assistance, and the staff at Springer-Verlag who guided me through the whole painful process. As usual, however, I retain sole responsibility for all the errors, mistakes, and other deviations from the truth that may be contained within these notes.

I would like to conclude by thanking the staff of the two computer centers where I did most of my numerical work (Cambridge and Brown). Without them these notes would never have existed, and they deserve far more praise than they ever seem to receive.

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