Geometric Theory of Dynamical Systems

An Introduction

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Translated by A. K. Manning

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Acknowledgments

This book grew from courses and seminars taught at IMPA and several other institutions both in Brazil and abroad, a first text being prepared for the Xth Brazilian Mathematical Colloquium. With several additions, it later became a book in the Brazilian mathematical collection *Projeto Euclides*, published in Portuguese. A number of improvements were again made for the present translation.

We are most grateful to many colleagues and students who provided us with useful suggestions and, above all, encouragement for us to present these introductory ideas on Geometric Dynamics. We are particularly thankful to Paulo Sad and, especially to Alcides Lins Neto, for writing part of a first set of notes, and to Anthony Manning for the translation into English.

Introduction

. . . cette étude qualitative (des équations différentielles) aura par elle-même un intérêt du premier ordre . . .

HENRI POINCARÉ, 1881.

We present in this book a view of the Geometric Theory of Dynamical Systems, which is introductory and yet gives the reader an understanding of some of the basic ideas involved in two important topics: structural stability and genericity.

This theory has been considered by many mathematicians starting with Poincaré, Liapunov and Birkhoff. In recent years some of its general aims were established and it experienced considerable development.

More than two decades passed between two important events: the work of Andronov and Pontryagin (1937) introducing the basic concept of structural stability and the articles of Peixoto (1958–1962) proving the density of stable vector fields on surfaces. It was then that Smale enriched the theory substantially by defining as a main objective the search for generic and stable properties and by obtaining results and proposing problems of great relevance in this context. In this same period Hartman and Grobman showed that local stability is a generic property. Soon after this Kupka and Smale successfully attacked the problem for periodic orbits.

We intend to give the reader the flavour of this theory by means of many examples and by the systematic proof of the Hartman–Grobman and the Stable Manifold Theorems (Chapter 2), the Kupka–Smale Theorem (Chapter 3) and Peixoto's Theorem (Chapter 4). Several of the proofs we give are simpler than the original ones and are open to important generalizations. In Chapter 4, we also discuss basic examples of stable diffeomorphisms with infinitely many periodic orbits. We state general results on the structural stability of dynamical systems and make some brief comments on other topics, like bifurcation theory. In the Appendix to Chapter 4, we present the important concept of rotation number and apply it to describe a beautiful example of a flow due to Cherry.

Prerequisites for reading this book are only a basic course on Differential Equations and another on Differentiable Manifolds the most relevant results of which are summarized in Chapter 1. In Chapter 2 little more is required than topics in Linear Algebra and the Implicit Function Theorem and Contraction Mapping Theorem in Banach Spaces. Chapter 3 is the least elementary but certainly not the most difficult. There we make systematic use of the Transversality Theorem. Formally Chapter 4 depends on Chapter 3 since we make use of the Kupka–Smale Theorem in the more elementary special case of two-dimensional surfaces.

Many relevant results and varied lines of research arise from the theorems proved here. A brief (and incomplete) account of these results is presented in the last part of the text. We hope that this book will give the reader an initial perspective on the theory and make it easier for him to approach the literature.

Rio de Janeiro, September 1981.

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Contents

List of Symbols

$\mathbb{R}$	real line
$\mathbb{R}^n$	Euclidean n-space
$\mathbb{C}^n$	complex n-space
C^n	differentiability class of mappings having n continuous derivatives
C^∞	infinitely differentiable
C^ω	real analytic
$df(p)$, df_p or $Df(p)$	derivative of f at p
$(\partial/\partial t)f$, $\partial f/\partial t$	partial derivative
$D_2 f(x, y)$	partial derivative with respect to the second variable
$d^n f(p)$	nth derivative of f at p
$L(\mathbb{R}^n, \mathbb{R}^m)$	space of linear mappings
$L^r(\mathbb{R}^m; \mathbb{R}^k)$	space of r-linear mappings
$\Vert\ \Vert$	norm
$g \circ f$	composition of the mappings g and f
$\emptyset$	empty set
$f \vert M$	restriction of map f to subset M
$\overline{U}$	closure of set U
TM_p	tangent space of M at p
TM	tangent bundle of M
$\mathfrak{X}^r(M)$	space of C^r vector fields on M
$f_* X$	vector field induced on the range of f by X
X_t	diffeomorphism induced by flow of X at time t
$\mathcal{O}(p)$	orbit of p
$\omega(p)$	ω-limit set of p
$\alpha(p)$	α-limit set of p
S^n	unit n-sphere

T^2	two-dimensional torus
grad f	gradient field of f
$\int f$	integral of f
id_M	identity map of M
$\langle\ ,\ \rangle$	Riemannian metric
$\langle\ ,\ \rangle_p$	inner product in the tangent space of p defined by Riemannian metric
$C^r(M, N)$	space of C^r mappings
$\|\ \|_r$	C^r-norm
$\mathrm{Diff}^r(M)$	space of C^r diffeomorphisms
$f \pitchfork S$	f is transversal to S
$\mathcal{O}_X(p)$	orbit of X through p
$\mathcal{O}_+(p)$	positive orbit of p
$\alpha'(t)$	derivative at t of map of interval
T^n	n-dimensional torus
$\mathscr{L}(\mathbb{R}^n)$	space of linear operators on $\mathbb{R}^n$
$\mathscr{L}(\mathbb{C}^n)$	complex vector space of linear operators on $\mathbb{C}^n$
L^k	$L \circ L \circ \cdots \circ L$
$\mathrm{Exp}(L)$, e^L	exponential of L
$GL(\mathbb{R}^n)$	group of invertible linear operators of $\mathbb{R}^n$
$H(\mathbb{R}^n)$	space of hyperbolic linear isomorphisms of $\mathbb{R}^n$
$\mathscr{H}(\mathbb{R}^n)$	space of hyperbolic linear vector fields of $\mathbb{R}^n$
$\mathrm{Sp}(L)$	spectrum of L
$\mathscr{G}_0$	space of vector fields having all singularities simple
$\det(A)$	determinant of A
$\mathscr{G}_1$	space of vector fields having all singularities hyperbolic
G_0	space of diffeomorphisms having all fixed points elementary
G_1	space of diffeomorphisms whose fixed points are all hyperbolic
$C_b^0(\mathbb{R}^m)$	space of continuous bounded maps from $\mathbb{R}^m$ to $\mathbb{R}^m$
$\dim M$	dimension of M
$W^s(p)$	stable manifold of p
$W^u(p)$	unstable manifold of p
$W^s_\beta(p)$	stable manifold of size β
$W^u_\beta(p)$	unstable manifold of size β
$W^s_{\mathrm{loc}}(0)$	local stable manifold
$W^u_{\mathrm{loc}}(0)$	local unstable manifold
$\mathscr{G}_{12}$	space of vector fields in $\mathscr{G}_1$ whose closed orbits are all hyperbolic
$\mathfrak{X}(T)$	space of vector fields in $\mathscr{G}_1$ whose closed orbits of period $\leq T$ are all hyperbolic
$L_\alpha(X)$	union of the α-limit sets of orbits of X
$L_\omega(X)$	union of the ω-limit sets of orbits of X
$\Omega(X)$	set of nonwandering points of X
M–S	set of Morse–Smale vector fields
∂M	boundary of M
int A	interior of set A