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Serge Lang

Complex Multiplication



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Preface

The small book by Shimura–Taniyama on the subject of complex multiplication is a classic. It gives the results obtained by them (and some by Weil) in the higher dimensional case, generalizing in a non-trivial way the method of Deuring for elliptic curves, by reduction mod p .

Partly through the work of Shimura himself (cf. [Sh 1] [Sh 2], and [Sh 5]), and some others (Serre, Tate, Kubota, Ribet, Deligne etc.) it is possible today to make a more snappy and extensive presentation of the fundamental results than was possible in 1961. Several persons have found my lecture notes on this subject useful to them, and so I have decided to publish this short book to make them more widely available.

Readers acquainted with the standard theory of abelian varieties, and who wish to get rapidly an idea of the fundamental facts of complex multiplication, are advised to look first at the two main theorems, Chapter 3, §6 and Chapter 4, §1, as well as the rest of Chapter 4. The applications of Chapter 6 could also be profitably read early.

I am much indebted to N. Schappacher for a careful reading of the manuscript resulting in a number of useful suggestions.

S. LANG

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