Maximum Principles in Differential Equations

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Maximum Principles in Differential Equations

With 56 Illustrations

Springer

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Mathematics Subject Classification: 35B50

This is a corrected reprint of the second printing of *Maximum Principles in Differential Equations* which was originally published in the Prentice-Hall Partial Differential Equations Series, © 1967 by Prentice-Hall, Englewood Cliffs, New Jersey.

Library of Congress Cataloging-in-Publication Data

Protter, Murray H.

Maximum principles in differential equations.

Reprint. Originally published: Englewood Cliffs, N.J., Prentice-Hall, 1967.

(Prentice-Hall partial differential equations series)

Bibliography: p.

Includes index.

1. Differential equations, Partial. 2. Maximum principles (Mathematics) I. Weinberger, Hans F. II. Title.

QA377.P76 1984 515.3'53 84-14028

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Softcover reprint of the hardcover 1st edition 1984

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Production managed by A. Orrantia; manufacturing supervised by Jacqui Ashri.

9 8 7 6 5 4 3 2 (Corrected second printing, 1999)

ISBN-13: 978-1-4612-9769-7

e-ISBN-13: 978-1-4612-5282-5

DOI: 10.1007/978-1-4612-5282-5

PREFACE

One of the most useful and best known tools employed in the study of partial differential equations is the maximum principle. This principle is a generalization of the elementary fact of calculus that any function $f(x)$ which satisfies the inequality $f'' > 0$ on an interval $[a, b]$ achieves its maximum value at one of the endpoints of the interval. We say that solutions of the inequality $f'' > 0$ satisfy a *maximum principle*. More generally, functions which satisfy a differential inequality in a domain D and, because of it, achieve their maxima on the boundary of D are said to possess a maximum principle.

The study of partial differential equations frequently begins with a classification of equations into various types. The equations most frequently studied are those of elliptic, parabolic, and hyperbolic types. Because equations of these three types arise naturally in many physical problems, mathematicians interested in partial differential equations have tended to concentrate their efforts on those developments which are of both mathematical and physical interest. A reader who learns differential equations by studying physically oriented problems not only parallels the historical development of the subject, but also acquires a clear understanding of the reasons some equations are studied in great detail while others are virtually ignored. Since many problems associated with equations of elliptic, parabolic, and hyperbolic types exhibit maximum principles, we feel that a study of the methods and techniques connected with these principles forms an excellent introduction or supplement to the study of partial differential equations.

There is usually a natural physical interpretation of the maximum principle in those problems in differential equations that arise in physics. In such situations the maximum principle helps us apply physical intuition to mathematical models. Consequently, anyone learning about the maximum principle becomes acquainted with the classically important partial differential equations and, at the same time, discovers the reasons for their importance.

The proofs required to establish the maximum principle are extremely elementary. By concentrating on those applications which can be derived from the maximum principle by elementary methods, we have been able to write this book at a level suitable for the undergraduate science student. Anyone who has completed a course in advanced calculus is qualified to read the entire book. In fact, any student who, in addition to elementary calculus, knows line integration, Green's theorem, and some simple facts on continuity and differentiation should find almost all of the book within his grasp.

The maximum principle enables us to obtain information about solutions of differential equations without any explicit knowledge of the solutions themselves. In particular, the maximum principle is a useful tool in the approximation of solutions, a subject of great interest to many scientists. This book should prove useful, not only to professional mathematicians and students primarily interested in mathematics, but also to those physicists, chemists, engineers, and economists interested in the numerical approximation of solutions of ordinary and partial differential equations and in the determination of bounds for the errors in such approximations.

The maximum principles for partial differential equations can be specialized to functions of one variable, and we have devoted the first chapter to a treatment of this one-dimensional case. The statement of the results and the proofs of the theorems are so simple that the reader should find this introduction to the subject strikingly easy. Of course, the one-dimensional maximum principle is related to second-order ordinary differential equations rather than to partial differential equations. In Chapter 1 we show that portions of the classical Sturm-Liouville theory are a direct consequence of the maximum principle. This chapter is included primarily because it provides an attractive and simple introduction to the various forms of the maximum principle which occur later. It also provides new ways of looking at some topics in the theory of ordinary differential equations.

In Chapter 2 we establish the maximum principle for elliptic operators, state several generalizations, and give a number of applications. Although the maximum principle for Laplace's and some other equations has been known for about a hundred years, it was relatively recently that Hopf established strong maximum principles for general second-order elliptic operators. Many of the important applications which we present make use of these results.

The maximum principle for parabolic operators takes a form quite different from that for elliptic operators. In Chapter 3 we present Nirenberg's strong maximum principle for parabolic operators. We then show, as in the elliptic case, that the principle may be used

to yield results on approximation and uniqueness. We conclude the chapter with a section on the maximum principle for a special class of parabolic systems.

The fourth and last chapter treats maximum principles for hyperbolic operators. The forms that these principles take reflect the structure of properly posed problems for hyperbolic equations. Both the statements of the theorems and the methods of proof for hyperbolic operators are quite different from those for elliptic and parabolic operators. In particular, the role of characteristic curves and surfaces becomes evident in the hyperbolic case.

The maximum principle occurs in so many places and in such varied forms that we have found it impossible to discuss some topics which we had originally hoped to treat. For example, the maximum principle for finite difference operators is omitted entirely. We do not mention the maximum principle for the modulus of an analytic function, a subject replete with important and interesting applications. Certain elliptic equations of order higher than the second are known to exhibit a maximum principle. (See, for example, Miranda [1] and Agmon [1].) We decided not to include this topic because advanced techniques of partial differential equations are needed.

Most of the notations and symbols we employ are fairly standard. A **domain** D in Euclidean space is an open connected set. The boundary of D is usually designated ∂D . The symbols $\cup$ and $\cap$ are used for the union and intersection of sets. Boldface letters denote vectors, and the customary notations u_{x_i} and $\partial u / \partial x_i$ are employed for partial derivatives.

We frequently use the letter L followed by brackets to denote a **linear operator** acting on functions. That is, L assigns to each function u of a certain class, a function $L[u]$ of another class. We say that L is **linear** if, whenever $L[u_1]$ and $L[u_2]$ are defined, the quantities $L[\alpha u_1 + \beta u_2]$ and $\alpha L[u_1] + \beta L[u_2]$ are also defined for all constants α and β , and the equation $L[\alpha u_1 + \beta u_2] = \alpha L[u_1] + \beta L[u_2]$ holds.

For those readers who may wish to explore the subject further, we have included at the end of each chapter a bibliographical discussion which contains historical references and a guide to other presentations and further results and applications relevant to the chapter. Since we have a continuing interest in the subject, we would enjoy hearing about results—new or old—which are related to the subject of this book.

We wish to thank the Air Force Office of Scientific Research and the National Science Foundation for their support of investigations leading to a number of results published here for the first time.

M. H. P.
H. F. W.

ACKNOWLEDGMENT

We are grateful to Springer-Verlag for making this reprint edition of our book available once again.

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