Undergraduate Texts in Mathematics

Undergraduate Texts in Mathematics

Apostol: Introduction to Analytic Number Theory.
Armstrong: Basic Topology.
Bak/Newman: Complex Analysis.
Banchoff/Wermer: Linear Algebra Through Geometry.
Brémaud: Introduction to Probability.
Childs: A Concrete Introduction to Higher Algebra.
Chung: Elementary Probability Theory with Stochastic Processes.
Croom: Basic Concepts of Algebraic Topology.
Curtis: Linear Algebra: An Introductory Approach.
Dixmier: General Topology.
Driver: Why Math?
Ebbinghaus/Flum/Thomas: Mathematical Logic.
Fischer: Intermediate Real Analysis.
Fleming: Functions of Several Variables. Second Edition.
Foulds: Optimization Techniques: An Introduction.
Foulds: Combination Optimization for Undergraduates.
Franklin: Methods of Mathematical Economics.
Halmos: Finite-Dimensional Vector Spaces. Second Edition.
Halmos: Naive Set Theory.
Iooss/Joseph: Elementary Stability and Bifurcation Theory.
James: Topological and Uniform Spaces.
Jänich: Topology.
Kemeny/Snell: Finite Markov Chains.
Klambauer: Aspects of Calculus.
Lang: A First Course in Calculus. Fifth Edition.
Lang: Calculus of Several Variables. Third Edition.
Lang: Introduction to Linear Algebra. Second Edition.
Lang: Linear Algebra. Third Edition.
Lang: Undergraduate Algebra.
Lang: Undergraduate Analysis.
Lax/Burstein/Lax: Calculus with Applications and Computing, Volume 1.
LeCuyer: College Mathematics with APL.
Lidl/Pilz: Applied Abstract Algebra.
Macki/Strauss: Introduction to Optimal Control Theory.
Malitz: Introduction to Mathematical Logic.
Marsden/Weinstein: Calculus I, II, III. Second Edition.
Martin: The Foundations of Geometry and the Non-Euclidean Plane.
Martin: Transformation Geometry: An Introduction to Symmetry.
Millman/Parker: Geometry: A Metric Approach with Models.
Owen: A First Course in the Mathematical Foundations of Thermodynamics.
Prenowitz/Jantosciak: Join Geometrics.
Priestly: Calculus: An Historical Approach.
Protter/Morrey: A First Course in Real Analysis. Second Edition.
Protter/Morrey: Intermediate Calculus.
Ross: Elementary Analysis: The Theory of Calculus.
Scharlau/Opolka: From Fermat to Minkowski.
Sigler: Algebra.
Simmonds: A Brief on Tensor Analysis.
Singer/Thorpe: Lecture Notes on Elementary Topology and Geometry.
Smith: Linear Algebra. Second Edition.
Smith: Primer of Modern Analysis.
Stanton/White: Constructive Combinatorics.
Thorpe: Elementary Topics in Differential Geometry.
Troutman: Variational Calculus with Elementary Convexity.
Wilson: Much Ado About Calculus.

I. M. James

Topological and Uniform Spaces

With 19 Illustrations

Springer-Verlag
New York Berlin Heidelberg
London Paris Tokyo

I. M. James
Mathematical Institute
University of Oxford
24–29 St. Giles
Oxford OX1 3LB
England

AMS Classifications: 54-01, 54A05, 54E15

Library of Congress Cataloging in Publication Data
James, I. M. (Ioan Mackenzie)
Topological and uniform spaces.
(Undergraduate texts in mathematics)
Bibliography: p.
Includes index.
1. Topology. 2. Topological spaces. 3. Uniform spaces. I. Title. II. Series.
QA611.J335 1987 514 86-28034

Typeset by Asco Trade Typesetting Ltd., Hong Kong.
Printed and bound by R. R. Donnelley & Sons, Harrisonburg, Virginia.

9 8 7 6 5 4 3 2 1

ISBN-13: 978-1-4612-9128-2 e-ISBN-13: 978-1-4612-4716-6
DOI: 10.1007/978-1-4612-4716-6

Contents

Introduction

This book is based on lectures I have given to undergraduate and graduate audiences at Oxford and elsewhere over the years. My aim has been to provide an outline of both the topological theory and the uniform theory, with an emphasis on the relation between the two. Although I hope that the prospective specialist may find it useful as an introduction it is the non-specialist I have had more in mind in selecting the contents. Thus I have tended to avoid the ingenious examples and counterexamples which often occupy much of the space in books on general topology, and I have tried to keep the number of definitions down to the essential minimum. There are no particular prerequisites but I have worked on the assumption that a potential reader will already have had some experience of working with sets and functions and will also be familiar with the basic concepts of algebra and analysis.

There are a number of fine books on general topology, some of which I have listed in the Select Bibliography at the end of this volume. Of course I have benefited greatly from this previous work in writing my own account. Undoubtedly the strongest influence is that of Bourbaki's *Topologie Générale* [2], the definitive treatment of the subject which first appeared over a generation ago. Although general topology has moved on since then, and alternative viewpoints have become important at the research level, it is remarkable how little there is in that volume which one would wish to see changed in any way. However, in a student text the exposition has to be constructed on different lines. I have reorganized the material, omitting topics which I felt were not of first importance for the non-specialist. I have tried to strengthen the motivation. Examples and diagrams are used to illustrate points at every suitable opportunity and each chapter except the preliminary one ends with a set of exercises.

The book divides naturally into three sections. Thus the first six chapters are devoted to the topological theory while the next two are devoted to the uniform theory. The last four, which are independent of each other, draw on ideas from both the topological section and the uniform section.

To avoid having to interrupt the course of the main exposition I have inserted a preliminary chapter, dealing with certain aspects of the theory of sets which will be required later. The topics discussed are somewhat miscellaneous in character. The first topic is the behaviour, for a given function, of the direct image and the inverse image in relation to the operations of complementation, union and intersection. This is followed by a discussion of the cartesian product. Relations are considered at various places in the main text, and so I thought it would be helpful to include a reminder of the notation and terminology used in connection with relations. Finally, and this is the most substantial part, I have given a brief outline of the theory of filters.

The first chapter of the topological section deals with the basic axioms. Illustrations are taken from interval topologies and metric topologies, with special reference to the real line. No previous knowledge of metric spaces is assumed.

The second chapter is concerned with continuity: topology is about continuous functions just as much as topological spaces. The topological product is dealt with here. Also topological groups are introduced at this stage both because of their intrinsic interest and because they provide such excellent illustrations of points in the general theory. Subspaces and quotient spaces are considered in Chapter 3, with a wide range of examples.

Most accounts of the theory go on to discuss separation axioms, connectedness and so forth at this point. But in my view compactness should come first, because of its fundamental importance. I believe the concept arises most naturally from a discussion of open functions and closed functions. This is not the orthodox approach, of course, but I have tried to justify it by showing that all the usual properties of compact spaces such as the Heine–Borel theorem can be proved quite simply and directly from this approach. I also show how compactness can be characterized in terms of filters, and incidentally show how the best-known characterization of compactness, in terms of open coverings, can be obtained. The general Tychonoff theorem is proved, followed by some observations on the subject of function spaces. This material occupies Chapters 4 and 5.

Chapter 6 is devoted to the separation axioms: the basic properties of Hausdorff, regular and normal spaces are established. In a later chapter there is an account of the corresponding functional separation axioms.

Chapter 7 of the uniform section deals with the basic axioms of uniform structure, with illustrations from topological groups and metric spaces. I have tried to show how the idea of a uniformity is a very natural one, in many ways more natural than the idea of a topology. This leads on to the notion of uniform continuity: the uniform theory is about uniformly continuous functions just as much as uniform spaces. I also deal with the uniform product with subspaces and, to a limited extent, with quotient spaces.

In Chapter 8 the connection between the uniform and the topological theories is established. It becomes clear at this stage that results about topological groups and metric spaces found earlier can be regarded as special cases of results about uniform spaces. The chapter continues with a discussion of the Cauchy condition, both for sequences and for filters. This lays the foundation for a subsequent chapter on completeness and completion.

The first of the last four chapters is concerned with connectedness. I show how connectedness, local connectedness and pathwise-connectedness are defined, for a topological space. I also discuss connectedness and local connectedness for uniform spaces.

The second of the last four chapters is concerned with countability. The first and second axioms are discussed, countable compactness and sequential compactness are considered, also the Lindelöf property and separability. In the next chapter we return to the separation axioms. After a glance at the functional Hausdorff property we discuss functional (i.e. complete) regularity. In particular, we show that complete regularity is a necessary and sufficient condition for a topological space to be uniformizable. We also prove Urysohn's theorem, to the effect that normal spaces are functionally normal.

The final chapter is concerned with completeness and completion, both for metric spaces and for uniform spaces. Metric completions are constructed both using the space of bounded continuous real-valued functions and via equivalence classes of Cauchy sequences. Then the uniform completion is constructed via equivalence classes of Cauchy filters.

Most branches of mathematics which have attained a certain degree of maturity have been developed as the result of the efforts of many different individuals. I have not included a historical section in my account, since the history of general topology is related in several of the other textbooks. I would, however, like to acknowledge my obvious debt to previous writers on the subject. I would also like to thank those who commented on various points which arose in the course of the work, particularly Dr. Alan Pears and Dr. Wilson Sutherland who kindly read an early draft and made a number of valuable suggestions; the former was also a great help at the proof stage.

Mathematical Institute
University of Oxford