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(continued after index)

Alexandru Dimca

Singularities and Topology of Hypersurfaces

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For Gabriela, John, George, and Maria

Preface

From the very beginning, algebraic topology has developed under the influence of the problems posed by trying to understand the topological properties of complex algebraic varieties (e.g., the pioneering work by Poincaré and Lefschetz). Especially in the work of Lefschetz [Lf2], the idea is made explicit that singularities are important in the study of the topology even in the case of smooth varieties.

What is known nowadays about the topology of smooth and singular varieties is quite impressive. The many existing results may be roughly divided into two classes as follows:

- (i) very general results or theories, like stratified Morse theory and (mixed) Hodge theory, see, for instance, Goresky–MacPherson [GM], Deligne [De1], and Steenbrink [S6]; and
- (ii) specific topics of great subtlety and beauty, like the study of the fundamental group of the complement in $\mathbb{P}^2$ of a singular plane curve initiated by Zariski or Griffiths’ theory relating the rational differential forms to the Hodge filtration on the middle cohomology group of a smooth projective hypersurface.

The aim of this book is precisely to introduce the reader to some topics in this latter class. Most of the results to be discussed, as well as the related notions, are at least two decades old, and specialists use them intensively and freely in their work. Nevertheless, it is impossible to find an adequate introduction to this subject, which gives a good feeling for its relations with other parts of algebraic geometry and topology.

The first three chapters of this book contain most of the prerequisites, as well as a broad survey of the local topology associated with a hypersurface singularity. The main purpose is to introduce the basic concepts and techniques, to motivate them, and to give natural and interesting examples. Many of the results in this first part are given without proofs, since there are excellent presentations of the local theory, see Arnold, Gusein-Zade, and Varchenko [AGV1], [AGV2], Looijenga [Lg], and Milnor [M5]. On the other hand, we prove here many “known” or “obvious” facts, for which it is

difficult to find a reference. Here is a detailed description of this first part of the book.

Chapter 1 starts with a presentation of the basic facts on regular Whitney stratifications, including their relations to μ^* -constant deformations and the first Thom isotopy lemma. The main output of this technique is the possibility of identifying families of singular algebraic varieties in which the topological type is constant, see (1.3.7)–(1.3.9). A short digression on the topology of affine hypersurfaces shows that singularities and stratifications may pop up even if we want to look only at the smooth fibers of a polynomial function (1.4.6). Then we discuss the local conic structure of an analytic set (1.5.1) and the cylindric structure at infinity of an affine algebraic set (1.6.9). We also state here a quite general version of a Zariski theorem of Lefschetz type (1.6.5) and point out the role played by Whitney stratifications. Here, as elsewhere in this book, it is not our purpose to state the most generally known version of this result, but a version that covers all the usual applications and that is also intuitively appealing.

Chapter 2 begins with a brief survey of classical knot theory, which can be regarded in many ways as a precursor of the modern study of the topology associated with a hypersurface singularity. Several key notions, like the group of a knot, the linking number (we even mention an apparently new definition for this numerical invariant in (2.1.18)), the Alexander polynomial, and the Seifert form, are introduced and illustrated by the simplest possible examples. Then we restrict our attention to the knots and links associated to plane curve singularities and discuss the main tools in their study: Puiseux parametrizations, Puiseux pairs, and iterated torus knots.

Next we make a digression and present the homology and cohomology Gysin sequences associated with a smooth hypersurface in a complex manifold (2.2.13)–(2.2.14) and the corresponding Poincaré–Leray residue morphisms. These Gysin sequences are used in (2.2.20) to describe the cohomology of the complement (in a small open ball) of a plane curve singularity in terms of meromorphic differential forms, as in Atiyah, Bott, and Garding [ABG]. This description is a prelude to the much more elaborate considerations in the same spirit as was done in Chapter 6.

The second half of Chapter 2 is devoted to surface singularities and their links. The main theme here is how the topological invariants of such a link can be computed from the resolution graph of the singularity, e.g., the classical theorem by Mumford that gives a presentation of the fundamental group of the link in terms of the resolution graph (2.3.8). We discuss in detail the topology of the links of the most frequently encountered classes of surface singularities, namely, the Hirzebruch–Jung singularities (2.4.1), the simple ADE singularities (2.4.3), the cusp singularities (2.4.5), the triangle singularities (2.4.7), the simple-elliptic singularities (2.4.9), and the weighted homogeneous singularities (2.4.10).

We start Chapter 3 with a discussion of the various fibrations known under the general heading of Milnor fibrations. In particular, we describe the

global affine Milnor fibration associated with a weighted homogeneous polynomial (3.1.11)–(3.1.14). Then we present the main results on the connectivity of the link of a hypersurface singularity, of the associated Milnor fiber, and of the boundary of this (compact) Milnor fiber (3.2.1)–(3.2.4) and (3.2.9). In the case of an isolated singularity, this boundary is diffeomorphic to the corresponding link, and maybe for this reason, this interesting manifold has not been investigated much until now (see (5.2.28) for a connection between such a boundary and the complement of a projective hypersurface). We compute, for a nonreduced hypersurface singularity, the number of connected components of its Milnor fiber (3.2.3).

Using miniversal deformations and morsifications, we next introduce the vanishing cycles and the Milnor lattice of an IHS (isolated hypersurface singularity). Special attention is devoted to the Thom–Sebastiani construction (3.3.19)–(3.3.23). This basic technical device is used several times in a crucial way in the sequel (e.g., in the proof of (4.4.16)).

In the final part of Chapter 3, we give necessary and sufficient conditions for the link of an IHS to be an integral or a rational homology sphere. In the case of the Brieskorn–Pham singularities, these conditions can be stated in terms of an associated graph. We give a proof of an elementary but elusive claim considered “obvious” in the original paper by Brieskorn [B1], see (3.4.13). These Brieskorn–Pham singularities are famous for at least two reasons:

- (i) Brieskorn [B1] used these singularities to show that an IHS singularity X can be a topological manifold when $\dim X \geq 3$. When $\dim X = 2$, such a phenomenon cannot occur because of Mumford’s theorem mentioned above (2.3.12).
- (ii) Hirzebruch [Hz2] discovered that some of the links associated with Brieskorn–Pham singularities are exotic spheres, i.e., manifolds homeomorphic but not diffeomorphic to the usual unit sphere.

We give a brief account of this second topic and note a formal analogy between one of the main formulas in this subject and a recent formula by Neumann–Wahl [NW] for the Casson invariant of certain links of surface singularities (3.4.15)–(3.4.18).

The final three chapters form the second and main part of this book. The underlying idea is to use the local topological information associated with a singularity, as described in the first part in order to compute some global topological invariants of complex algebraic varieties (usually hypersurfaces in a projective or affine space). Most of the results in this part of the book are given with complete proofs, and examples and exercises continue to abound.

Chapter 4 deals with the computation of the fundamental group of a hypersurface complement in a projective space. Some simple results even treat the case of a hypersurface complement in a weighted projective space, using a result due to Armstrong (4.1.15). Please note that basic information on weighted projective spaces and on their subvarieties is collected in

Appendix B for the reader's convenience. After some generalities, the problem of computing such fundamental groups is reduced to the case of curve complements in the plane via a Zariski theorem of Lefschetz type (4.1.17). This is a classical and difficult subject started by Zariski in his famous paper [Z1].

Before attacking this problem, we introduce, following an idea of Libgober, the Alexander polynomials for a projective hypersurface (4.1.18)–(4.1.26). As in classical knot theory, these polynomials are invariants easier to compute than the homotopy groups of the corresponding hypersurface complements. There is also a preliminary discussion on presentations of groups. Here we just want to fix the notation and to introduce the reader to some basic noncommutative groups, namely, the binary dihedral groups $\tilde{D}_k$ (4.2.1(iii)), the free product of two groups (4.2.1(iv)), and some braid groups (4.2.6)–(4.2.13). Using a basic result by Serre about groups acting on trees, we give a “topological” construction for the free product $(\mathbb{Z}/p\mathbb{Z}) * (\mathbb{Z}/q\mathbb{Z})$ for $(p, q) = 1$. We recall then the usual version (with an open cover) of the van Kampen theorem (4.2.17), but we also remark that what is sometimes needed in the sequel is a very sophisticated version of it (with a closed covering satisfying a long list of conditions!).

Next we investigate an affine (or local) problem, namely, the computation of the fundamental group of the complement $\mathbb{C}^2 \setminus C$, where C is the affine plane curve given by the equation

$$C: x^p - y^q = 0.$$

Here we follow a paper by Oka [O5], but make some of the details more explicit. In this way, we get a presentation for the group $\pi_1(\mathbb{C}^2 \setminus C)$ in which the relations are the prototypes for the so-called “monodromy relations” to be introduced later. Then we discuss the Zariski–van Kampen theorem that gives a presentation for the fundamental group of a plane curve complement in terms of generators and monodromy relations (4.3.15). Several instances where this fundamental group is abelian are described, namely, when the curve is smooth, when the curve has an inflection point of order equal to the degree (4.3.17), when the curve has only nodes as singularities (4.1.13) and (4.3.18), or when the curve is irreducible and has just one singular point with high multiplicity (4.3.8), (4.3.11), and (4.3.21). In the last section of Chapter 4, we discuss some noncommutative fundamental groups of plane curve complements. The main two examples treated here come from the original paper by Zariski [Z1], namely, the three cuspidal quartic curve (4.4.8) and the sextic curve with six cusps situated on a conic (4.4.16). In fact, in (4.4.16) we treat a more general example due to Oka, but with a different idea of proof based on Némethi [N2] and on Serre's result mentioned above. At the end we give probably the first example of an irreducible curve having just one singularity and still with a noncommutative associated fundamental group (4.4.21).

In Chapter 5 we consider mainly the integral homology and cohomology of projective complete intersections. We start this discussion with some facts on the topology of a complex projective space and on the line bundles on such

a space. Then we discuss the cohomology properties of a projective complete intersection, making no assumption on its singular locus (5.2.6), (5.2.11), (5.2.12), and (5.2.17). We present a criterion for a hypersurface to have the same rational homology groups as a projective space (5.2.22) and give some examples of this situation. To each (singular) projective complete intersection, one can associate a smooth manifold, namely, the boundary of a good “tubular” neighborhood. In the hypersurface case, we relate this manifold to the boundary of the corresponding Milnor fiber (5.2.28) and to the hypersurface complement (5.2.31).

Next we consider in detail the cohomology structure of a smooth projective complete intersection (5.3.1), (5.3.6), and (5.3.7). Using the Hirzebruch Index Theorem (5.3.11), we can easily compute the signature of an even-dimensional complete intersection. The cup product on the middle cohomology group can then be determined, once we know its parity, via a method due to Libgober–Wood (5.3.16). We end our discussion of the smooth case by comparing the topological classification to the diffeomorphism classification for the two-dimensional smooth complete intersections (5.3.31)–(5.3.34). For this discussion we follow closely Ebeling [E4].

The final section of Chapter 5 deals with projective complete intersections with isolated singularities and is based on my paper [D2]. The main result (5.4.3) relates the integral homology of such a complete intersection to the Milnor lattices of its singularities. For the reader’s convenience, we have collected some useful facts on integer bilinear forms, discriminant groups, Dynkin diagrams, and related notions in Appendix A. As an illustration of this technique, we compute the homology of all the normal cubic surfaces in $\mathbb{P}^3$ in (5.4.8), as well as for some hypersurfaces introduced by Barthel and this author [BD] that have the same integral homology like a projective space (5.4.12) and (5.4.13). Last we discuss some related topics, namely, the homology of some projective cones (5.4.18)–(5.4.21) and the cup product on the middle cohomology group (5.4.24)–(5.4.27). We end this chapter with an example (5.4.28) showing the difficulty of understanding the topology of affine hypersurfaces by compactifying them. We use this also as an opportunity to present to the reader a useful classical trick from algebraic topology (related to the Smith theory), see (5.4.31) and (5.4.34).

The last chapter essentially aims at computing the Betti numbers of a singular projective hypersurface using rational differential forms defined on its complement. Most of the results in this chapter are also valid for hypersurfaces in weighted projective spaces. We start with a discussion of the rational differential forms defined on a hypersurface complement and find out that such a form can be written quite explicitly (6.1.16) and (6.1.17). Then we recall two basic results by Grothendieck that say that the de Rham cohomology of an affine (or Stein) hypersurface complement can be computed using rational (or polar meromorphic) differential forms (6.1.21) and (6.1.22). These special differential forms induce in a natural way a polar filtration on the cohomology of such a complement. This cohomology has also a Hodge filtra-

tion as part of its mixed Hodge structure (basic information on mixed Hodge structures is collected in Appendix C). A fundamental fact due to Deligne and this author [DD] is that these two filtrations are related by an inclusion in the projective case (6.1.31). In the local case, results by Karpishpan and this author show that the situation is similar, but slightly more complicated (6.1.39) and (6.1.40). We present next some spectral sequences that converge to the cohomology of a hypersurface complement, in the same spirit as in my papers [D5], [D8], and [D9]. When the hypersurface under consideration is smooth, this approach goes back to Griffiths' fundamental paper [G]. In this smooth case, we give examples (6.2.18) that show some of the advantages of having a method to produce explicit bases for the cohomology groups of hypersurface complements. We concentrate then on the first difficult case, namely, to understand the topology associated to a weighted homogeneous polynomial f such that the affine hypersurface $\{f = 0\}$ has a one-dimensional singular locus. With every irreducible component of this singular locus, one can associate a transversal singularity, which is in fact an isolated hypersurface singularity together with a finite cyclic group of automorphisms. There is a conjectural formula for the Euler characteristics of the weighted hypersurface and of the Milnor fiber associated to the polynomial f (which surely holds in most of the interesting cases) involving only the weights and the degree of the polynomial f , as well as some local invariants coming from the transversal singularities, see (3.19) in [D5] and relevant parts of [D8].

In this book we treat the more difficult problem of computing the corresponding Betti numbers and Alexander polynomials. It is well known that these topological invariants depend not only on such local invariants of the transversal singularities, but also on the position of these singularities, see Zariski [Z1]. To understand this strange behavior, we derive a useful exact sequence (6.3.15) in which everything can be, at least in principle, computed explicitly in terms of differential forms. The efficacy of our method is then tested on several examples (which had been considered by several authors using different methods, based essentially on the resolution of singularities), namely, nodal hypersurfaces (6.4.5), sextic curves with six cusps (6.4.9) and some of their higher dimensional generalizations (6.4.14), and two configurations of nine lines introduced by Artal-Bartolo (6.4.15). As another application we give a sufficient condition for the triviality of the Alexander polynomial of a hypersurface with an unique singularity (6.4.17).

Finally, in this chapter, we point out that for a hypersurface with only isolated singularities the usual Betti numbers and the intersection Betti numbers (i.e., the dimensions of the middle perversity intersection cohomology groups) determine each other by very simple formulas (6.4.22) and (6.4.23).

This book is designed for a large audience of graduate students and professional mathematicians. Most of it is addressed to workers in algebraic geometry and algebraic and differential topology. However, there are also parts for

those with an interest in pure algebra (e.g., the group-theoretic discussion in Chapter 4 or Appendix A) or in complex manifolds (parts of Chapters 5 and 6).

This book can serve a graduate student either as additional material to a general course in algebraic or differential topology (giving natural and interesting examples that are missing from many textbooks on these subjects!), or as new motivation for a student dealing with some abstract parts of algebraic geometry (e.g., mixed Hodge structures). The open questions raised in the text may stimulate the student's imagination in his or her work toward a doctorate. There are plenty of exercises to give additional information and to help the reader to test his or her understanding of the material.

As in any book devoted to a subject that has evolved over a century at least, it was not possible to avoid many references to results needed, but not proved, in this book. Generally, we have avoided the long and technically complicated proofs and offered the reader the shorter and more illuminating ones (when they are available). The reference system in the book uses two numbers, e.g., (3.7) to refer to a result within the same chapter; and three numbers, e.g., (4.3.7) to refer to a result in a different chapter (whose number appears then in the first position).

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Sydney, Australia

Alexandru Dimca

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