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# Models of Phase Transitions

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# Preface

... "What do you call work?"  
"Why ain't that work?"  
Tom resumed his whitewashing, and answered carelessly:  
"Well, maybe it is, and maybe it ain't. All I know, is, it suits Tom Sawyer."  
"Oh come, now, you do not mean to let on that you like it?"  
The brush continued to move.  
"Like it? Well, I do not see why I oughtn't to like it. Does a boy get a chance to whitewash a fence every day?"  
That put the thing in a new light. Ben stopped nibbling the apple. ...  
(From Mark Twain's *Adventures of Tom Sawyer*, Chapter II.)

Mathematics can put quantitative phenomena in a new light; in turn applications may provide a vivid support for mathematical concepts.

This volume illustrates some aspects of the mathematical treatment of phase transitions, namely, the classical Stefan problem and its generalizations. The intended reader is a researcher in application-oriented mathematics. An effort has been made to make a part of the book accessible to beginners, as well as physicists and engineers with a mathematical background. Some room has also been devoted to illustrate analytical tools.

This volume deals with research I initiated when I was affiliated with the Istituto di Analisi Numerica del C.N.R. in Pavia, and then continued at the Dipartimento di Matematica dell'Università di Trento. It was typeset by the author in plain T<sub>E</sub>X.

I express my gratitude to E. Magenes, who introduced me to the Stefan problem; to A. Damlamian, who helped me at the beginning of that research; to S. Luckhaus, to whom I owe some of the ideas which are developed here in Chap. VIII; to P. Colli, N. Kenmochi, and P. Krejčí, who kindly read some parts of the draft and contributed several suggestions.

I am also indebted to H. Brézis, who included this volume in the series he directs; and to the staff of Birkäuser, who assisted me in the redaction of this work.<sup>(1)</sup>

Povo di Trento, August 1996



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<sup>(1)</sup> *Work?* Does a researcher get a chance to write a book every day?