THE BOOK OF *Numbers*

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Preface

The Book of Numbers seems an obvious choice for our title, since its undoubted success can be followed by *Deuteronomy*, *Joshua*, and so on; indeed the only risk is that there may be a demand for the earlier books in the series. More seriously, our aim is to bring to the inquisitive reader without particular mathematical background an explanation of the multitudinous ways in which the word "number" is used. We have done this in a way which is free from the formality of textbooks and syllabuses, so that the professional mathematician can also glean important information in areas outside her own speciality, and correspondingly enrich her teaching.

The uses of the word number are diverse, but we can identify at least three separate strands. The development of number (usually written in the singular), continually adapting and generalizing itself to meet the needs of both mathematics and its increasing variety of applications: the counting numbers, zero, fractions, negative numbers, quadratic surds, algebraic numbers, transcendental numbers, infinitesimal and transfinite numbers, surreal numbers, complex numbers, quaternions, octonions.

Then there is the special study of the integers, Gauss's higher arithmetic, or the theory of numbers (usually written in the plural), which overlaps the more recent area of enumerative combinatorics. Special sets or sequences of numbers: the prime numbers, Mersenne and Fermat numbers, perfect numbers, Fibonacci and Catalan numbers, Euler and Eulerian numbers, Bernoulli numbers.

Finally there is a host of special numbers: Ludolph's π, Napier's e, Euler's γ, Feigenbaum's constant, algebraic numbers which arise in specific contexts ranging from the diagonal of a square or of a regular pentagon to the example of degree 71 which arises from the apparently simple "look and say" sequences.

Although mathematics is traditionally arranged in logical sequences, that is not the way the human brain seems to work. While it is often useful to know parts of earlier chapters in the book when reading some of the later ones, the reader can browse at will, picking up nuggets of information, with no obligation to read steadily from cover to cover.

Chapter 1 describes number words and symbols and Chapter 2 shows how many elementary but important facts can be discovered "without using any mathematics." Chapters 3 and 4 exhibit several sets of whole numbers which can manifest themselves in quite different contexts, while Chapter 5 is devoted to the "multiplicative building blocks," the prime numbers. Fractions, or "algebraic numbers of degree one," are dealt with in Chapter 6 and those of higher degree in Chapter 7. So-called "complex" and "transcendental" numbers are explained in Chapters 8 and 9. The final chapter is devoted to the infinite and the infinitesimal, to surreal numbers which constitute an extremely large, yet infinitely small, subclass of the most recent development of number, the values of combinatorial games.

The color graphics in Chapter 2 are the work of Kenny and Andy Guy, using Brian Wyvill's Graphicsland software at the Graphics Laboratory in the Computer Science Department at The University of Calgary. Three of the figures in Chapter 4 are reproduced from Przemyslaw Prusinkiewicz and Aristid Lindenmayer, *The Algorithmic Beauty of Plants*, Springer-Verlag, New York 1990, by kind permission of the authors and publishers. The first figure is due to D.R. Fowler, the second to D.R.F. and P. Prusinkiewicz and the third to D.R.F. and A. Snider.

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Princeton and Calgary John H. Conway and Richard K. Guy

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